The Hilbert 3/2 Structure and Weil-Petersson Metric on the Space of the Diffeomorphisms of the Circle Modulo Conformal Maps.

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Abstract

The space of diffeomorphisms of the circle modulo boundary values of conformal maps of the disk. The Sobolev 3/2 norm on the tangent space space at *id* of the space of diffeomorphisms of the circle modulo boundary values of conformal maps of the disk induces the only Kähler left invariant metric defined up to a constant on the space of diffeomorphisms of the circle modulo boundary values of conformal maps of the disk. We proved before that the completion of the holomorphic tangent space at the identity of the space of diffeomorphisms of the circle modulo boundary values of conformal maps of the disk with respect to the Sobolev 3/2 norm can be embedded as a closed Hilbert susbspace into the tangent space at a point of the Segal-Wilson Grassmannian. It was established in our previous paper that the natural metric on the Segal-Wilson Grassmannian

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is Kähler has a negative sectional curvature in the holomorphic direction bounded away from zero and the exponential map is a complex analytic biholomorphic map from the tangent space of a given point of the Segal-Wilson Grassmannian to the Segal-Wilson Grassmannian. In this paper we prove that the completion of the space of diffeomorphisms of the circle modulo boundary values of conformal maps of the disk with respect to the left invariant Kähler metric is complex analytically isomorphic to the closed totally geodesic closed Hilbert submanifold in the Segal-Wilson Grassmannian using the exponential map. The new proof is much simpler and avoid the technical proof that the change of the coordinates is complex analytic.

Contents

1 Introduction

1.1 General Remarks

The recent developments in string theory and the study of the theory of infinite dimensional completely integrable systems renew the interest in the study of differential geometric properties of infinite dimensional complex manifolds. Donaldson in [5] and [6] proposed the study of the existence of canonical metrics on certain complex manifolds by the study of the differential geometry of some infinite dimensional manifolds.

The geometric aspects of the space $\mathbf{T}^{\infty} := \mathbf{Diff}_{+}^{\infty}\left(S^{1}\right)/\mathbb{PSU}_{1,1}$, where $\mathbf{Diff}_{+}^{\infty}\left(S^{1}\right)$ is the group of C^{∞} diffeomorphisms of the circle preserving the orientations and, $\mathbb{PSU}_{1,1}$ is the group of conformal maps of the unit disk restricted to its boundary were studied in [19]. It is a well known fact that there exists a unique up to constant left invariant Kähler metric on $\mathbf{T}^{\infty}$. In [19] we prove that the holomorphic curvature of the left invariant Kähler metric on $\mathbf{T}^{\infty}$ is negative. We prove that the completion of $\mathbf{T}^{\infty}$ with respect to the left invariant Kähler metric defines Hilbert complex manifold structure on $\mathbf{T}^{\infty}$ modeled by the Sobolev $3/2$ norm.

In this paper we give a new proof of one of the main Theorems in [19]:

Theorem. *The completion $\mathbf{T}^{3/2}$ of $\mathbf{T}^{\infty}$ with respect to the left invariant Kähler metric defines Hilbert complex manifold structure on $\mathbf{T}^{\infty}$ modeled by the Sobolev $3/2$ norm.*

The proof presented here is much simpler than the proof we gave in [19]. Our new proof is purely differential geometric. It is based on two facts. We can embed the tangent space $T_{id,\mathbf{T}^{\infty}}$ at $id \in \mathbf{T}^{\infty}$ into the tangent space at some point isomorphic to the Hilbert space of Hilbert-Schmidt operators onto $\mathbf{T}^{\infty}$ of the Segal-Wilson Grassmanian $\mathbb{G}\mathbf{r}_{\infty}$ isometrically. We observed that the exponential map defined by the left invariant Kähler metric on $\mathbb{G}\mathbf{r}_{\infty}$ is a complex analytic diffeomorphism from the tangent space at any point onto $\mathbb{G}\mathbf{r}_{\infty}$. Thus $\mathbf{T}^{3/2}$ is the image of the completion of $T_{id,\mathbf{T}^{\infty}}$ in the tangent space at some point of $\mathbb{G}\mathbf{r}_{\infty}$ under the exponential map. Thus we avoid the technical check that the transition functions of one open set to another are complex analytic functions.

In this paper we simplified also the proof of the embedding Theorem given in [19]. We gave a necessary and sufficient condition for the boundary values on the unit circle of a holomorphic function to have $-3/2$ finite Sobolev norm. This fact is simple and helps to simplify the proof of the embedding Theorem.

A Theorem of Kuiper states that any Hilbert manifold is isomorphic to an open set in a Hilbert space. We believe that our proof of the compatibilities of the unique Kähler metric and the complex analytic Hilbert manifold modeled after $3/2$ Sobolev, norm is very natural and simple in view of the above Theorem of Kuiper.

Some of the results in the paper [20] are closely related to our results obtained in [19].

We have included an Appendix in which we gave proofs of the two main results obtained in [19] about the properties of the left invariant metric that we

use in the paper. This is done to make the paper completely self contained.

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1.2 Outline of the Proof of the Existence of Sobolev 3/2 Hilbert Complex Structure on $\mathbf{T}^\infty$

The new proof of that the completion of $\mathbf{T}^\infty$ with respect to the left invariant Kähler metric defines Hilbert complex manifold structure on $\mathbf{T}^\infty$ modeled by the Sobolev $3/2$ norm is based on three facts:

Fact 1. Let $\mu_\psi(z) := \left(1-|z|^2\right)^2 \psi(\overline{z})$, $\psi(\overline{z})$ is an antiholomorphic function on the unit disk $\mathbb{D}$, and

$$\psi(\overline{z})|_{S^1} \in \mathbf{H}^{-3/2}_{S^1,h}, \tag{1}$$

where $\mathbf{H}^{-3/2}_{S^1,h}$ is the space of the restrictions of holomorphic functions on the unit circle with finite $-3/2$ Sobolev norm. Then we have

$$\sup_{|z|<1} |\mu_\psi(z)| = \|\psi(\overline{z})\|_{\mathbf{L}^\infty(\mathbb{D})} < C_\psi < \infty. \tag{2}$$

This fact was proved also independently and very elegantly in [20].

Fact 2. Let us define

$$\mathbb{H}^{-1,1}(\mathbb{D}) :=$$

$$\left\{\mu_\psi(z) \left| |||\mu_\psi(z)|||_{\mathbf{L}^\infty_{\mathbb{D}}} < \infty,\ \partial\psi(\overline{z}) = 0,\ \ \&\ \ \psi(\overline{z})|_{S^1} \in \mathbf{H}^{-3/2}_{S^1}\right.\right\}. \tag{3}$$

We will define a Hilbert structure on $\mathbb{H}^{-1,1}(\mathbb{D})$. It will be based on the following non degenerate $\mathbf{L}^2$ pairing on the unit disk $\mathbb{D}$, $\mathbb{H}^{-1,1}(\mathbb{D}) \times \mathbf{H}^{-3/2}_{S^1,h} \to \mathbb{C}$ given by

$$\langle \mu_\psi(z), \eta(\overline{z})\rangle|_{S^1} = \frac{\sqrt{-1}}{2\pi}\int_{\mathbb{D}} \left(1-|z|^2\right)^2 \psi(\overline{z})(z)\overline{\eta(\overline{z})}dz \wedge \overline{dz} =$$

$$\frac{\sqrt{-1}}{2\pi}\int_{\mathbb{D}} \mu_\psi(z)\overline{\eta(z)}dz \wedge \overline{dz}. \tag{4}$$

Direct computations show

$$\left|\frac{\sqrt{-1}}{2\pi}\int_{\mathbb{D}} \mu_\psi(z)\overline{\eta(z)}dz \wedge \overline{dz}\right| = \left|\langle\, \psi(\overline{z})|_{S^1}, \eta(\overline{z})|_{S^1}\rangle_{\mathbf{H}^{-3/2}_{S^1,h}}\right| < \infty.$$

$\mathbb{H}^{-1,1}(\mathbb{D})$ is the $\mathbf{L}^2$ dual to $\mathbf{H}^{-3/2}_{S^1,h}$ by the pairing (4) and thus it is isomorphic to $\mathbf{H}^{3/2}_{S^1,h}$. We will give an explicit isomorphism between $\mathbb{H}^{-1,1}(\mathbb{D})$ and $\mathbf{H}^{3/2}_{S^1,h}$.

Let

$$\psi\left(\overline{z}\right)=\sum_{n\geq 2}a_n\overline{z}^{n-2} \text{ and } \Psi\left(\frac{1}{z}\right)=\sum_{n\geq 2}\frac{a_n}{n(n+1)(n+2)}\left(\frac{1}{z}\right)^{n-2}. \tag{5}$$

It is easy to see that (1) implies

$$\Psi\left(\frac{1}{z}\right)\Big|_{S^1}\in\mathbf{H}^{3/2}_{S^1} \tag{6}$$

Consider the map $F:\mathbb{H}^{-1,1}\left(\mathbb{D}\right)\rightarrow\mathbf{H}^{3/2}_{S^1}$ given by

$$F\left(\mu_\psi(z)\right)=\Psi\left(\frac{1}{z}\right)\Big|_{S^1}. \tag{7}$$

We define a Hilbert norm of $\mathbb{H}^{-1,1}\left(\mathbb{D}\right)$ as follows:

$$\|\mu_\psi(z)\|^2_{\mathbb{H}^{-1,1}(\mathbb{D})}=\|\Psi(z)\|^2_{\mathbf{H}^{3/2}_{S^1}}. \tag{8}$$

We will show that completion of the tangent space $T_{id,\mathbf{T}^\infty}$ with respect to the left invariant metric is isomorphic to the Hilbert space $\mathbb{H}^{-1,1}\left(\mathbb{D}\right)$ with the norm defined by (8).

This observation has the following interpretation; The space $\mathbf{H}^{-3/2}_{S^1,h}$ can be identified with a quadratic differentials

$$\psi\left(\frac{1}{z}\right)\left(d\left(\frac{1}{z}\right)\right)^{\otimes 2}$$

which is isomorphic to completion of the holomorphic cotangent space $\Omega^{1,0}_{\mathbf{T}^\infty}$ with respect to the left invariant Kähler metric when $\psi\left(\frac{1}{z}\right)\big|_{S^1}\in\mathbf{H}^{-3/2}_{S^1,h}$. $\mathbb{H}^{-1,1}\left(\mathbb{D}\right)\approxeq\mathbf{H}^{3/2}_{S^1,h}$ can be identified with the completion of the tangent space $T_{id,\mathbf{T}^\infty}$ and the duality is given by the contraction of $\psi\left(\frac{1}{z}\right)\left(d\left(\frac{1}{z}\right)\right)^{\otimes 2}$ with the Poincare metric on $\mathbb{D}$. The Poincare metric induces the left invariant Kähler metric on $\mathbf{T}^\infty$. Thus the space $\mathbb{H}^{-1,1}\left(\mathbb{D}\right)$ consists of harmonic Beltrami differentials $\mu_\psi\left(\overline{dz}\otimes\frac{d}{dz}\right)$ with respect to left invariant Kähler metric.

Fact 3. In this paper we give a new and simpler proof of the following fact proved in [19]; The map

$$\iota:\mu_\psi(z)\rightarrow\ker\left(\overline{\partial}-\mu_\psi\partial\right)$$

where μ_ψ is such that $\|\mu_\psi\|_{\mathbf{L}^\infty(\mathbb{D})}\leq k<1$ is an embedding into $T_{\mathbf{H}^+,\mathbb{G}r_\infty}$. From here we can conclude that $\iota\left(\mathbb{H}^{-1,1}\left(\mathbb{D}\right)\right)$ is closed Hilbert subspace in the tangent space $T_{\mathbf{H}^+,\mathbb{G}r_\infty}$ which is the Hilbert space of Hilbert-Scmidt operators $T:\mathbf{H}^+\rightarrow\mathbf{H}^-$. $\mathbf{H}^+(\mathbf{H}^-)$ are the space of the boundary values holomorphic (antiholomorphic) functions in the unit disk with a finite $\mathbf{L}^2(S^1)$ norm. From

here we can conclude that $\iota\left(\mathbb{H}^{-1,1}(\mathbb{D})\right)$ is closed Hilbert subspace in the tangent space $T_{\mathbf{H}^{+},\mathbb{G}r_{\infty}}$.

We observed in [19] that the left invariant metric on $\mathbb{G}r_{\infty}$ is Kähler and it has a negative holomorphic sectional curvature bounded away from zero. From this fact we derived that the exponential map

$$\exp : T_{\mathbf{H}^{+},\mathbb{G}r_{\infty}} \rightarrow \mathbb{G}r_{\infty}$$

is a complex analytic diffeomorphism.

Nag proved that $\mathbf{T}^{\infty}$ together with the left invariant Kähler metric can be embedded isometrically into $\mathbb{G}r_{\infty}$. Thus we can conclude that the left invariant Kähler metric on $\mathbf{T}^{\infty}$ modeled by the Sobolev $3/2$ space of $T_{id,\mathbf{T}^{\infty}}$ has a negative holomorphic sectional curvature bounded away from zero. The result of Nag implies that the completion $\mathbf{T}^{3/2}$ of $\mathbf{T}^{\infty}$ with respect to the left invariant Kähler metric on $\mathbf{T}^{\infty}$ will be complex analytically isomorphic to the totally geodesic complex analytic Hilbert submanifold $\exp\left(\iota\left(\mathbb{H}^{-1,1}(\mathbb{D})\right)\right)$ in $\mathbb{G}r_{\infty}$.

2 Preliminary Material

2.1 Basic Notations and Definitions

Notation 2 *Let* $\mathbb{D} := \{z \in \mathbb{C} \,||z| < 1\}$ *and* $\mathbb{D}^{*} := \left\{\frac{1}{z} \in \mathbb{C} \,||z| < 1\right\}$. *We recall that the Schwarzian derivative of an analytic function* f *is defined by*

$$[\mathcal{S}(f)](z) \overset{def}{=} \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^{2}.$$

Definition 3 *Let* $\psi(\overline{z})$ *be antiholomorphic function in the unit disk. Suppose that* $\sup\limits_{z\in\mathbb{D}}\left|\left(1-|z|^{2}\right)^{2}\psi(\overline{z})\right| \leq k < 1$. *Define*

$$\mu_{\psi}(z) = \begin{cases} \left(1-|z|^{2}\right)^{2}\psi(\overline{z}), & z \in \mathbb{D} \\ 0, & z \notin \mathbb{D} \end{cases}. \tag{9a}$$

Then by abuse of notation define the Beltrami differential

$$\mu_{\psi} := \mu_{\psi}(z)\left(\overline{dz} \otimes \frac{\partial}{\partial z}\right).$$

Notation 4 *We will call the operator* $\overline{\partial}_{\mu_{\psi}} := \overline{\partial} - \mu_{\psi}(z)\partial$ *Beltrami operator. This space we will denote it by* $\mathbb{H}^{1,1}_{1}(\mathbb{D})$.

We will used the following Theorem of Ahlfors and Weill stated bellow. The proof can be found in page 100 of [7]:

Theorem 5 *Let $\Psi(z):\mathbb{D}^*\to\overline{\mathbb{C}}$ be an univalent function that can be extended to a quasi-conformal map of the Riemann sphere $\mathbb{CP}^1$. Let*

$$\Psi(z)=z+\sum_{k=1}^{\infty}a_k\left(\frac{1}{z}\right)^k,\ \mathcal{S}\left[\Phi(z)|_{\mathbb{D}^*}\right]=\sum_{k\geq 4}^{\infty}b_k(z)^{-k} \qquad (10)$$

for $|z|<1$. Let

$$\mu_\psi(z)=\begin{cases} -\frac{1}{2}\left(1-|z|^2\right)^2\frac{\mathcal{S}[\Phi(z)|_{\mathbb{D}^*}](\overline{z})}{z^4}, & |z|<1 \\ 0, & |z|\geq 1\end{cases} \qquad (11)$$

Then, for the quasi-conformal mapping Φ_{μ_ψ} on the Riemann sphere satisfying

$$\left(\overline{\partial}-\mu_\psi(z)\partial\right)\Phi_{\mu_\psi}(z)=0$$

we have that $\psi(z)=\mathcal{S}\left[\Phi(z)|_{\mathbb{D}^*}\right]$ and $\Phi_{\mu_\psi}(z)\big|_{\mathbb{D}^*}=\Psi(z)$ is the univalent function on $\mathbb{D}^*$ defined by (10).

Definition 6 *We will define: $\mathbf{T}^\infty:=\mathbf{Diff}_+^\infty\left(S^1\right)/\mathbb{PSU}_{1,1}$.*

Definition 7 *Let M be an even dimensional C^∞ manifold, T be the tangent bundle and T^* be the cotangent bundle on M. We will say that M has an almost complex structure if there exists a section $I\in C^\infty(M,Hom(T^*,T^*))$ such that $I^2=-id$.*

This definition is equivalent to the following one:

Definition 8 *Let M be an even-dimensional C^∞ manifold. If $T_M^*\otimes\mathbb{C}=\Omega_M^{1,0}\oplus\overline{\Omega_M^{1,0}}$ is a global splitting of the complexified cotangent bundle such that $\Omega_M^{0,1}=\overline{\Omega_M^{1,0}}$, then we will say that M has an almost complex structure.*

Definition 9 *We will say that an almost complex structure is an integrable one, if for each point $x\in M$ there exists an open set $U\subset M$ such that we can find local coordinates $z^1,..,z^n$ such that $dz^1,..,dz^n$ are linearly independent at each point $m\in U$ and they generate $\Omega^{1,0}|_U$.*

Notation 10 *We define:*

$$\mathbf{H}^+=\left\{f(z)\in H(\mathbb{D})|\ f(z)=\sum_{n\geq 0}a_nz^n\ \&\ \|f(z)\|^2=\sum_{n\geq 0}|a_n|^2<\infty\right\} \qquad (12)$$

and

$$\mathbf{H}^-=\left\{g(z)\in H(\mathbb{D}^*)|\ f(z)=\sum_{n\geq 1}b_nz^{-n}\ \&\ \|g(z)\|^2=\sum_{n\geq 0}|b_n|^2<\infty\right\}. \qquad (13)$$

Definition 11 *We define $\mathbb{H}^{\alpha}_{S^1,h}$ to be the space of all holomorphic functions $g(z)$ on $\mathbb{D}^*$ such that the restriction of $g(z)$ on the boundary S^1 of $\mathbb{D}^*$ is an element of the Hilbert space with a Sobolev α norm, i.e.*

$$\|g\|^2_{\alpha} = \sum_{n=0}^{\infty} n^{2\alpha} |b_n|^2 < \infty,$$

where α is any real number.

Remark 12 *In this paper, for convenience, we shall use the following notation to the norms $3/2$ and $-3/2$, which are equivalent to the Sobolev $3/2$ and $-3/2$ norms:*

$$\|g\|^2_{3/2} := \sum_{n=2}^{\infty} n\left(n^2-1\right)|a_n|^2 \text{ and } \|g\|^2_{-3/2} := \sum_{n=2}^{\infty} \frac{|a_n|^2}{n\left(n^2-1\right)}.$$

Remark 13 *Let $\psi_i(\theta) = \sum_{k\geq 2}^{\infty} a^i_k e^{-i(k-2)\theta} \in \mathbb{H}^{p/q}_{S^1,h}$ be two complex valued functions on S^1. The $\mathbf{L}^2_{S^1}$ pairing between $\mathbb{H}^{p/q}_{S^1,h}$ and $\mathbb{H}^{-p/q}_{S^1,h}$ is defined by*

$$\langle \psi_1, \psi_2 \rangle = \sum_{k=2}^{\infty} a^1_k \overline{a^2_k}. \tag{14}$$

The pairing (14) is a non-degenerate pairing.

2.2 Left Invariant Kähler Structures on $\mathbf{Diff}^{\infty}_{+}(S^1)/\mathbb{PSU}_{1,1}$

The complexified tangent space $T_{id,\mathbf{Diff}^{\infty}_{+}(S^1)/\mathbb{PSU}_{1,1}}$ at the identity can be described as follows:

$$T_{id,\mathbf{Diff}^{\infty}_{+}(S^1)/\mathbb{PSU}_{1,1}} \otimes \mathbb{C} = \left\{ f(\theta)\frac{d}{d\theta} \,\middle|\, f(\theta) = \sum_{n\neq 0,\pm 1} a_n e^{in\theta} \right\}.$$

Thus we have a splitting:

$$T_{id,\mathbf{Diff}^{\infty}_{+}(S^1)/\mathbb{PSU}_{1,1}} \otimes \mathbb{C} = T^{1,0}_{id,\mathbf{Diff}^{\infty}_{+}(S^1)/\mathbb{PSU}_{1,1}} \oplus \overline{T^{1,0}_{id,\mathbf{Diff}^{\infty}_{+}(S^1)/\mathbb{PSU}_{1,1}}},$$

where

$$T^{1,0}_{id,\mathbf{Diff}^{\infty}_{+}(S^1)/\mathbb{PSU}_{1,1}} = \left\{ f(\theta)\frac{d}{d\theta} \,\middle|\, f(\theta) = \sum_{n>1} c_n e^{i(n-2)\theta} \right\}.$$

The complexified tangent space $T_{id,\mathbf{Diff}^{\infty}_{+}(S^1)/\mathbb{PSU}_{1,1}}$ at the identity is presented as follows

$$T_{id,\mathbf{Diff}^{\infty}_{+}(S^1)/\mathbb{PSU}_{1,1}} = T^{1,0}_{id,\mathbf{Diff}^{\infty}_{+}(S^1)/\mathbb{PSU}_{1,1}} \oplus \overline{T^{1,0}_{id,\mathbf{Diff}^{\infty}_{+}(S^1)/\mathbb{PSU}_{1,1}}}.$$

Thus a left invariant almost complex structure on the coadjoint orbit and $\mathbf{T}^{\infty} := \mathbf{Diff}_{+}^{\infty}(S^1)/\mathbb{PSU}_{1,1}$ was defined. It is proved in [16] that this complex structure is integrable.

We will define the analogue of the Weil-Petersson metric on $\mathbf{T}^{\infty}$. It is proved in [9], [3], [22] and [21] that the Weil-Petersson metric is unique left invariant Kähler metric defined up to a constant on $\mathbf{T}^{\infty}$.

Definition 14 *Let*

$$f(\theta)\frac{d}{d\theta} = \left(\sum_{n>1} a_n e^{i(n-2)\theta}\right)\frac{d}{d\theta} \in T^{1,0}_{id,\mathbf{Diff}_{+}^{\infty}(S^1)/\mathbb{PSU}_{1,1}}.$$

Then the norm $\left\|f(\theta)\frac{d}{d\theta}\right\|_{W.P.}$ *of* $f(\theta)\frac{d}{d\theta}$ *is given by*

$$\left\|f(\theta)\frac{d}{d\theta}\right\|^2_{W.P.} = \sum_{n>1}\frac{n\left(n^2-1\right)}{12}\left|a_n\right|^2. \tag{15}$$

3 Preliminary Results

3.1 $\mathbf{H}^{-3/2}_{S^1,h}$ and the $\mathbf{L}^{\infty}$ Norm

Theorem 15 *Let* $\mathbf{H}^{-3/2}_{S^1,h}$ *be the Hilbert space defined by Definition 11. Let* $|z\} \leq 1$. *Suppose that* $\psi(z) = \sum_{n=2}^{\infty} a_n\left(\frac{1}{z}\right)^{n-2}$ *is a holomorphic function defined on* $\mathbb{D}^*$ *such that* $\psi(z)|_{S^1} \in \mathbf{H}^{-3/2}_{S^1,h}$. *Let* $\psi(\overline{z}) = \sum_{n=2}^{\infty} a_n(\overline{z})^{n-2}$. *Then* $\left(1-|z|^2\right)^2|\psi(\overline{z})| < C$ *for* $z \in \mathbb{D}$, *i.e.* $|z| < 1$.

Proof: The proof of Theorem 15 can be find in [20]. We will repeat the proof given in [20]. The condition $\psi(z)|_{S_1} \in \mathbf{H}^{-3/2}_{S^1,h}$ means that we have

$$\|\psi\|^2_{\mathbf{H}^{-3/2}_{S^1,h}} := \sum_{n\geq 2}\frac{|a_n|^2}{n\left(n^2-1\right)} < \infty\ . \tag{16}$$

Let $b_n := \frac{a_n}{\sqrt{n(n^2-1)}}$. Then Cauchy-Schwarz inequality implies that

$$|\psi(\overline{z})| \leq \left|\sum_{n=2}^{\infty} n\left(n^2-1\right)|b_n|^2\right|^{1/2}\left|\sum_{n=2}^{\infty} n\left(n^2-1\right)|\overline{z}|^{2n-4}\right|^{1/2} \tag{17}$$

for every $z \in \mathbb{D}$. Since

$$\left|\sum_{n=2}^{\infty} n\left(n^2-1\right)|\overline{z}|^{2n-4}\right| = \frac{6}{\left(1-|z|^2\right)^2} \tag{18}$$

we get that (17) and (18) imply that $\left(1-|z|^2\right)^2|\psi(\overline{z})|\leq C$. So Theorem 15 is proved. ■

Theorem 16 *Suppose that* $\psi(\overline{z})=\sum_{n=2}^{\infty}a_n\overline{z}^n$ *is a holomorphic function defined on* $\mathbb{D}$ *such that* $\psi(\overline{z})|_{S^1}\in\mathbf{H}^{-3/2}_{S^1,h}$. *Then for some real* α *such that* $0<\alpha<1$ *we have*

$$\lim_{\substack{|z|\to 1\\ |z|<1}}\frac{|\psi(|\overline{z}|)|}{(1-|z|)^{a}}=C_\psi<\infty \tag{19}$$

Proof: Since $\psi(\overline{z})$ is an atiholomorphic function, then the maximum principle shows that the function $\max_{r=|z|<1}|\psi(|z|)|=\psi(r)$ is an increasing function. Suppose that $\lim_{r\to\infty}\psi(r)=\infty$. Theorem 20 implies that

$$\left(1-|r|^2\right)^2\psi(r)=(1-|r|)^2(1+|r|)^2\psi(r)<C.$$

So we get that

$$\lim_{r\to\infty}\frac{\psi(r)}{(1-|r|)^{\alpha}}=C_\psi<\infty \tag{20a}$$

for some $0<\alpha<2$. Easy computations in polar coordinates show that

$$\int_{\mathbb{D}}\left(1-|z|^2\right)^2|\psi(\overline{z})|^2\,dz\wedge\overline{dz}=\|\psi(\overline{z})|_{S^1}\|^2_{\mathbf{H}^{-3/2}_{S^1,h}}.$$

Thus the assumption that $\psi(\overline{z})|_{S^1}\in\mathbf{H}^{-3/2}_{S^1,h}$ implies that

$$\left|\int_{\mathbb{D}}\left(1-|z|^2\right)^2|\psi(\overline{z})|^2\,dz\wedge\overline{dz}\right|<\infty \tag{21}$$

Using polar coordinates (21) implies that

$$\int_0^1((1-|r|)(1+|r|)|\psi(r)|)\,dr<4\int_0^1((1-|r|)|\psi(r)|)\,dr<\infty.$$

The last inequality and (20a) imply that

$$\lim_{0<r<1,\ r\to 1}\sup_{|z|=|r|}|\psi(z)|(1-r)^{-\alpha}=C>0$$

for $0<\alpha<1$. Theorem 16 is proved. ■

Corollary 17 *Let* $\psi(\overline{z})|_{S^1}\in\mathbf{H}^{-3/2}_{S^1,h}$. *Then*

$$\left(1-|z|^2\right)^2\psi(\overline{z})\left(\frac{d}{dz}\otimes\overline{dz}\right)\Big|_{\mathbb{D}^*}=0. \tag{22}$$

3.2 Duality

Theorem 18 *Let* $\psi(\overline{z}) = \sum_{n=2}^{\infty} a_n (\overline{z})^{n-2}$ *and* $\psi(\overline{z})|_{S^1} \in \mathbf{H}^{-3/2}_{S^1,h}$*. Let us define the linear functional* L_ψ *on* $\mathbf{H}^{-3/2}_{S^1,h}$ *as follows:*

$$L_\psi(\eta(z)) := \frac{1}{2\pi i}\int_{\mathbb{D}} \left(1-|z|^2\right)^2 \psi(\overline{z})\overline{\eta(\overline{z})}dz \wedge \overline{dz}. \tag{23}$$

where $\eta(z)$ *is a holomorphic function on* $\mathbb{D}^*$ *such that* $\eta(z)|_{S^1} \in \mathbf{H}^{-3/2}_{S^1,h}$*. Then*

1. *The linear functional* $L_\psi(\eta(z))$ *defined by* (23) *is continuous,*
2. *The map*

$$\psi(z)|_{S^1} = \psi(\overline{z})|_{S^1} \in \mathbf{H}^{-3/2}_{S^1,h} \to L_\psi \in \mathbf{H}^{3/2}_{S^1,} = Hom\left(\mathbf{H}^{-3/2}_{S^1,h}, \mathbb{C}\right)$$

is a canonical isomorphism between $\mathbf{H}^{-3/2}_{S^1,h}$ *and its dual* $\mathbf{H}^{3/2}_{S^1,h}$ with respect to *the* $\mathbf{L}^2_{S^1}$.
3. Let $\Psi(\overline{z})$ *be defined by* (5)*. Then we have the following formula:*

$$L_\psi(\eta(z)) = \langle \Psi(\theta), \eta(\theta)\rangle_{\mathbf{L}^2(S^1)} = \langle \psi(\theta), \eta(\theta)\rangle_{\mathbf{H}^{-3/2}_{S^1,h}}. \tag{24}$$

The map $\psi \to \Psi$ *defines an isomorphism between* $\mathbf{H}^{3/2}_{S^1,h}$ *and* $Hom\left(\mathbf{H}^{-3/2}_{S^1,h}, \mathbb{C}\right)$.

Proof: Now we will proceed with proof of Theorem 18.

Lemma 19 *Let* $|z| < 1$*. Let* $\psi(z) = \sum_{n>1} a_n \left(\frac{1}{z}\right)^{n-2}$ *be a holomorphic function in* $\mathbb{D}^*$ *such that* $\psi(z)|_{S^1} \in \mathbf{H}^{-3/2}_{S^1,h}$*. Let us define* $\Psi(\overline{z})$ *by* (5)*. Then*

$$\Psi(\overline{z})|_{S^1} = \Psi(\theta) \in \mathbf{H}^{3/2}_{S^1,h} \;\&\; \|\Psi\|^2_{\mathbf{H}^{3/2}_{S^1,h}} = \|\psi\|^2_{\mathbf{H}^{-3/2}_{S^1,h}}.$$

Proof: Let $\psi(\overline{z}) = \sum_{n\geq 2}^{\infty} a_n (\overline{z})^{n-2}$ be the antiholomorphic function in the unit disk $\mathbb{D}$ defined by the coefficients a_n of $\psi(z)$ by . The assumption $\psi(z)|_{S^1} \in \mathbf{H}^{-3/2}_{S^1,h}$ implies that

$$\|\psi\|^2_{\mathbf{H}^{-3/2}_{S^1,h}} = \sum_{n=2}^{\infty} \frac{|a_n|^2}{n(n^2-1)} < \infty.$$

The definition (5) of $\Psi(\overline{z}) = \sum_{n=0}^{\infty} b_n (\overline{z})^n$ states that the coefficients b_n of $\Psi(\overline{z})$ are given by $b_n = \frac{a_n}{n(n^2-1)}$. From here we get that Sobolev 3 /2 norm of $\Psi(\overline{z})|_{S^1}$ is given by

$$\|\Psi\|^2_{\mathbf{H}^{3/2}_{S^1,h}} = \sum_{n=2}^{\infty} \left(n\left(n^2-1\right)\right)|b_n|^2 = \sum_{n=2}^{\infty} \frac{n\left(n^2-1\right)|a_n|^2}{\left(n\left(n^2-1\right)\right)^2} =$$

$$\sum_{n=2}^{\infty}\frac{|a_n|^2}{n\,(n^2-1)}=\|\Psi(z)\|^2_{\mathbf{H}^{3/2}_{S^1,h}}=\|\psi(z)\|^2_{\mathbf{H}^{-3/2}_{S^1,h}}<\infty.$$

Lemma 19 is proved. ■

Lemma 20 *Let $\psi_i\,(z)$ be two holomorphic functions in $\mathbb{D}^*$ such that*

$$\psi_i\,(z)\,|_{S^1}=\psi_i(\theta)=\sum_{n=2}^{\infty}a_n^i e^{-2\pi i(n-2)\theta}\in\mathbf{H}^{-3/2}_{S_1,h}.$$

Let $\psi_i(\overline{z})=\sum_{n=2}^{\infty}a_n^i\,(\overline{z})^{n-2}$ be the anti holomorphic functions on $\mathbb{D}$ such that $\psi_i\,(\overline{z})\,|_{S^1}=\psi_i(\theta)$. Then the following formulas hold:

$$\langle\psi_1(\theta),\psi_2(\theta)\rangle_{\mathbf{H}^{-3/2}_{S^1,h}}:=\frac{1}{2\pi}\int\limits_{\mathbb{D}}(1-|z|^2)^2\psi_1(\overline{z})\overline{\psi_2(\overline{z})}dz\wedge\overline{dz}<\infty \tag{25}$$

Proof: The definition of the Hardy space $\mathbf{H}^{-3/2}_{S_1,h}$ implies

$$\psi_i\,(z)\in\mathbf{H}^{-3/2}_{S_1,h}\Longleftrightarrow\|\psi_i\|^2_{\mathbf{H}^{-3/2}_{S^1,h}}=\sum_{n=2}^{\infty}\frac{\left|a_n^i\right|^2}{n\,(n^2-1)}<\infty \tag{26}$$

for $i=1$ and 2. By the definition of the scalar product in the Hilbert space $\mathbf{H}^{-3/2}_{S^1,h}$ we have

$$\left\langle\psi_1(\theta),\overline{\psi_2(\theta)}\right\rangle_{\mathbf{H}^{-3/2}_{S^1,h}}=\sum_{n=2}^{\infty}\frac{a_n^1\overline{a_n^2}}{n\,(n^2-1)}. \tag{27}$$

Direct computations show that

$$\frac{1}{2\pi}\int\limits_{\mathbb{D}}(1-|z|^2)^2\psi_1(\overline{z})\overline{\psi_2(\overline{z})}dz\wedge\overline{dz}=$$

$$\int\limits_0^1\left(1-r^2\right)^2\left(\sum_{n=2}^{\infty}a_n^1\overline{a_n^2}r^{2n-1}\right)dr=\sum_{n=2}^{\infty}\frac{a_n^1\overline{a_n^2}}{n\,(n^2-1)}. \tag{28}$$

The facts that $\psi_1\,(z)$ & $\psi_2\,(z)\,|_{S^1}\in\mathbf{H}^{-3/2}_{S_1,h}$ and Cauchy-Schwarz inequality imply that the power series in (28) converges. Comparing formula (28) with (27) we derive formula (25). Lemma 20 is proved. ■

Lemma 21 *Let $\psi(z)$ be a holomorphic function in $\mathbb{D}$ such that $\psi\,(z)\in\mathbf{H}^{-3/2}_{S_1,h}$. Then for each holomorphic function $\eta\,(z)$ on $\mathbb{D}^*$ such that $\eta\,(z)\,|_{S^1}\in\mathbf{H}^{-3/2}_{S_1}$ and $\|\eta\|^2_{\mathbf{H}^{-3/2}_{S^1,h}}=1$, we have*

$$\max_{\|\eta\|^2_{\mathbf{H}^{-3/2}_{S^1,h}}=1}|L_\psi\,(\eta)|=\langle\Psi\,(\theta)\,,\psi(\theta)\rangle_{\mathbf{L}^2(S^1)}=\|\psi\|^2_{\mathbf{H}^{-3/2}_{S^1,h}}=\|\Psi\|^2_{\mathbf{H}^{3/2}_{S^1,h}}. \tag{29}$$

Proof: (26), (28), the definition $L_\psi(\eta(\theta)) = \langle \psi, \eta \rangle_{\mathbf{H}^{-3/2}_{S^1,h}}$ and Cauchy-Schwarz inequality imply that

$$|L_\psi(\eta(\theta))| = \left| \langle \psi, \eta \rangle_{\mathbf{H}^{-3/2}_{S^1,h}} \right|^2 \leq \|\psi\|^2_{\mathbf{H}^{-3/2}_{S^1,h}} \|\eta\|^2_{\mathbf{H}^{-3/2}_{S^1,h}} .$$

Since we assumed that $\|\eta\|^2_{\mathbf{H}^{-3/2}_{S^1,h}} = 1$ then

$$|L_\psi(\eta(\theta))| = \left| \langle \psi, \eta \rangle_{\mathbf{H}^{-3/2}_{S^1,h}} \right|^2 \leq \|\psi\|^2_{\mathbf{H}^{-3/2}_{S^1,h}} .$$

Let $\eta = \frac{\psi}{\|\psi\|}$. So we have

$$\left| L_\psi \left(\frac{\psi}{\|\psi\|}(\theta) \right) \right| = \left| \left\langle \psi, \frac{\psi}{\|\psi\|} \right\rangle_{\mathbf{H}^{-3/2}_{S^1,h}} \right|^2 = \|\psi\|^2_{\mathbf{H}^{-3/2}_{S^1,h}} .$$

The definition (3) of $\Psi(z)$ implies that

$$\| \Psi(z)|_{S^1} \|^2_{\mathbf{H}^{3/2}_{S^1,h}} = \sum_{n \geq 1} \frac{|a_n|^2}{n(n+1)(n+2)} = \|\psi\|^2_{\mathbf{H}^{-3/2}_{S^1,h}} .$$

Lemma 21 is proved. ■

According to Lemma 21 we have

$$\max_{\|\psi_2\|^2_{\mathbf{H}^{-3/2}_{S^1,h}} = 1} |L_{\psi_1}(\psi_2(\theta))| = \|\psi_1\|^2_{\mathbf{H}^{-3/2}_{S^1,h}} .$$

Formula (25), the expressions for $\Psi(\theta)$ given by (5) and $\eta(\overline{z})\,|_{S^1} \in \mathbf{H}^{-3/2}_{S_1,h}$ imply that

$$\langle \Psi(\theta), \eta(\theta) \rangle_{\mathbf{L}^2(S^1)} = \langle \psi, \eta \rangle_{\mathbf{H}^{-3/2}_{S^1,h}} = \sum_{n>1} \frac{a_n \overline{b_n}}{n(n^2-1)}. \tag{30}$$

This fact implies that the linear functional L_ψ defined by (23) is continuous on $\mathbf{H}^{-3/2}_{S^1,h}$. Thus $L_\psi \in \left(\mathbf{H}^{-3/2}_{S^1,h}\right)^* = \mathbf{H}^{3/2}_{S^1,h}$ and the map

$$\psi \in \mathbf{H}^{-3/2}_{S^1,h} \to L_\psi \in \left(\mathbf{H}^{-3/2}_{S^1,h}\right)^* = \mathbf{H}^{3/2}_{S^1,h} \tag{31}$$

is linear and continuous. Formula (28) implies that since we can choose $\psi_1 = \psi_2$, the map (31) has zero kernel. Part **1** and **2** of Theorem 18 are proved. ■

Let $\eta(\overline{z})$ be any antiholomorphic function on $\mathbb{D}$ such that

$$\eta(\overline{z})\,|_{S^1} = \eta(\theta) = \sum_{n>1} b_n e^{-i(n-2)\theta} \in \mathbf{H}^{-3/2}_{S_1,h} .$$

Formula (25), the expressions for $\Psi(\theta)$ given by (5) and $\eta(\overline{z})|_{S^1} \in \mathbf{H}^{-3/2}_{S_1,h}$ imply that

$$\langle\Psi(\theta),\eta(\theta)\rangle_{\mathbf{L}^2(S^1)} = \langle\psi,\eta\rangle_{\mathbf{H}^{-3/2}_{S^1,h}} = \sum_{n>1}\frac{a_n\overline{b_n}}{n(n^2-1)}. \tag{32}$$

Comparing (25) and (32) we get that

$$\langle\psi(\theta),\eta(\theta)\rangle_{\mathbf{H}^{-3/2}_{S^1,h}} = \int_{\mathbb{D}}(1-|z|^2)^2\psi(\overline{z})\overline{\eta(\overline{z})}dz\wedge\overline{dz} = \langle\Psi(\theta),\eta(\theta)\rangle_{\mathbf{L}^2(S^1)}. \tag{33}$$

Thus part **3** of Theorem 18 is proved. ■ Theorem 18 is proved. ■

3.3 Geometric Interpretation of the Duality

The motivation for the proof of Theorem 22 is the generalization of the following fact about the finite dimensional Teichmüller Theory to the infinite dimensions.

Let R be a Riemann surface of genus g greater than, or equal to two. Let g_0 be the metric with constant curvature in R. Let $\mathbf{T}_g$ be the Teichmüller space of all Riemann surfaces of genus g. It is a well known fact that the holomorphic tangent space $T_{\tau_R,\mathbf{T}_g}$ is isomorphic to the space $\mathbb{H}^1\left(R,T^{1,0}_R\right)$ of harmonic $(0,1)$ forms with coefficients in the holomorphic tangent bundle with respect the metric of constant curvature on R. The cotangent space $\Omega^{1,0}_{\tau_R,\mathbf{T}_g}$ is isomorphic to the space $H^0\left(R,\left(\Omega^{1,0}_R\right)^{\otimes 2}\right)$ of holomorphic quadratic differentials. See [1].

The Serre duality $\left(H^0\left(R,\left(\Omega^{1,0}_R\right)^{\otimes 2}\right)\right)^* = \mathbb{H}^1\left(R,T^{1,0}_R\right)$, is given explicitly as follows; Let $z=x+iy$,

$$\psi(z)\,(dz)^{\otimes 2}\in H^0\left(R,\left(\Omega^{1,0}_R\right)^{\otimes 2}\right)$$

and $\frac{4dx\otimes dy}{(1-x^2-y^2)^2}$ be the metric with constant curvature on $\mathbb{D}$, which is the universal cover of R. Then

$$\overline{\left(\psi(z)\,(dz)^{\otimes 2}\right)\lrcorner\left(\left(1-|z|^2\right)^2\frac{d}{dz}\otimes\overline{\frac{d}{dz}}\right)} =$$

$$\left(1-|z|^2\right)^2\overline{\psi(z)}\left(\overline{dz}\otimes\frac{d}{dz}\right)\in\mathbb{H}^1\left(R,T^{1,0}_R\right).$$

will be a harmonic element of $\mathbb{H}^1\left(R,T^{1,0}_R\right)$ with respect to g_0.

Theorem 22 *The completion of $T_{id,\mathbf{T}^\infty}$ with respect to the left invariant Kähler metric on $\mathbf{T}^\infty$ is a Hilbert space $(T_{id,\mathbf{T}^\infty})^{3/2}$ isomorphic to to $\mathbf{H}^{3/2}_{S^1,h}$. The completion of $\Omega^{1,0}_{id,\mathbf{T}^\infty}$ with respect to the left invariant Kähler metric on $\mathbf{T}^\infty$ is a Hilbert space $\left(\Omega^{1,0}_{id,\mathbf{T}^\infty}\right)^{-3/2}$ isomorphic to to $\mathbf{H}^{-3/2}_{S^1,h}$.*

Proof: The proof of Theorem 22 follows from the following Lemma:

Lemma 23 *1. The holomorphic cotangent bundle $\Omega^1_{id,\mathbf{T}^\infty}$ at $id \in \mathbf{T}^\infty$ is canonically isomorphic to the space*

$$\left\{ \psi\left(\overline{z}\right)\left(dz\right)^{\otimes 2} \middle| \, \psi\left(\overline{z}\right) = \sum_{n\geq 2} a_n \left(\overline{z}\right)^n \ \ \textit{and}\ |z|<1 \right\}, \tag{34}$$

where $\psi\left(\overline{z}\right)|_{S^1}$ is a C^∞ function. 2. The holomorphic tangent space $T_{id,\mathbf{T}^\infty}$ at $id \in \mathbf{T}^\infty$ can be identified with all Beltrami differentials

$$(1-|z|^2)^2\psi\left(\overline{z}\right)\left(\overline{dz}\otimes\frac{d}{dz}\right) = \mu_\psi\left(\overline{dz}\otimes\frac{d}{dz}\right), \tag{35}$$

where $\psi\left(\overline{z}\right)|_{S^1}$ is a C^∞.

Proof of part 1: According to [17] the space $\mathbf{T}^\infty$ can be viewed as the deformation space of all complex structures on the unit disk $\mathbb{D}$. The holomorphic cotangent bundle $\Omega^1_{id,\mathbf{T}^\infty}$ at $id \in \mathbf{T}^\infty$ can be identified with the space of all holomorphic quadratic differentials

$$\psi\left(\frac{1}{z}\right)\left(d\left(\frac{1}{z}\right)\right)^{\otimes 2} = z^4\psi\left(\frac{1}{z}\right)\left(dz\right)^{\otimes 2}, \tag{36}$$

where $\psi\left(\frac{1}{z}\right)$ is the Schwarzian of a univalent function $\Phi\left(\frac{1}{z}\right)$ on the complement of the unit disk. This follows from Theorem 5. So we have

$$\psi\left(\frac{1}{z}\right) = \sum_{n\geq 4} a_n\left(\frac{1}{z}\right)^n$$

and it is a complex analytic function in $\mathbb{C}-\mathbb{D}=\mathbb{D}^*$ such that $\psi\left(\frac{1}{z}\right)\big|_{S^1}$ is a C^∞. We can canonically continue $\psi\left(\frac{1}{z}\right)\big|_{S^1}$ to an antiholomorphic function

$$\psi\left(\overline{z}\right) = \sum_{n>0} a_n\left(\overline{z}\right)^n \tag{37}$$

in $\mathbb{D}$. Thus (36) and (37) imply (34). ■

Proof of part 2: According to [21] the left invariant Kähler metric on $\mathbf{T}^\infty$ is defined as follows on $\Omega^1_{id,\mathbf{T}^\infty}$:

$$\frac{\sqrt{-1}}{2\pi}\int_{\mathbb{D}}\psi_1\left(\overline{z}\right)\left(dz\right)^{\otimes 2}\overline{\psi_2\left(\overline{z}\right)}\left(\overline{dz}\right)^{\otimes 2}\left(1-|z|^2\right)^2\frac{d}{dz}\wedge\overline{\frac{d}{dz}} =$$

$$\frac{\sqrt{-1}}{2\pi}\int_{\mathbb{D}}\left(1-|z|^2\right)^2\psi_1\left(\overline{z}\right)\overline{\psi_2\left(\overline{z}\right)}dz\wedge\overline{dz}.$$

Since we know that the metric gives a canonical isomorphism between $\Omega^1_{id,\mathbf{T}^\infty}$ the conjugate of the tangent space $T_{id,\mathbf{T}^\infty}$ we derive that$T_{id,\mathbf{T}^\infty}$ can be identified with all Beltrami differentials

$$(1-|z|^2)^2\psi\left(\overline{z}\right)\left(\overline{dz}\otimes\frac{d}{dz}\right)=\mu_\psi\left(\overline{dz}\otimes\frac{d}{dz}\right),$$

where $\psi\left(\overline{z}\right)|_{S^1}$ are C^∞ functions. So (35) is proved. ■

Theorem 18 and Lemma 23 imply Theorem 22. ■

Corollary 24 *Let $|z|<1$, $\psi(z)=\sum\limits_{n=2}a_n\left(\frac{1}{z}\right)^{n-2}$ be a holomorphic function on $\mathbb{D}^*$. We already proved that the quadratic differential $\psi(z)d\left(\frac{1}{z}\right)^{\otimes 2}\in\Omega^{1,0}_{id,\mathbf{T}^\infty}$. Then the Weil-Petersson norm of the quadratic differential $\psi(z)d\left(\frac{1}{z}\right)^{\otimes 2}$ with respect to the left invariant Kähler metric is given by*

$$\|\psi\|^2_{W.P.}=L_\psi\left(\psi\right)=\left\langle\Psi\left(\theta\right),\psi\left(\theta\right)\right\rangle_{\mathbf{L}^2(S^1)}=\|\psi\left(\theta\right)\|^2_{\mathbf{H}^{-3/2}_{S^1}}.$$

Corollary 25 *Let $\mu_\psi\in\mathbb{H}^{-1,1}(\mathbb{D})$, where $\mathbb{H}^{-1,1}(\mathbb{D})$ is defined by (3). Then the map $\digamma$ defined by (7) is surjective isomorphism between the Hilbert spaces $\mathbb{H}^{-1,1}(\mathbb{D})$ and $\mathbf{H}^{3/2}_{h,S^1}$*

Proof: The definition (3) of $\mathbb{H}^{-1,1}(\mathbb{D})$, the definition (5) and the definition (7) of the map $\digamma$ directly imply that the map $\digamma$ is surjective one to one. Part 1 of Corollary 25 is proved. ■

Remark 26 *In the paper [20] on page 23 it is stated "In this section we are going to endow $T(1)$ (the universal Teichmüller space" with a structure of a complex manifold modeled on the separable Hilbert space*

$$A_2(\mathbb{D}):=\left\{\phi\ hol\ in\ \mathbb{D}:\|\phi\|^2_2=\iint\limits_{\mathbb{D}}|\phi|^2\left(1-|z|^2\right)^2dz^2\right\}$$

*of holomorphic function on D. In the corresponding topology, the universal Teichmüller space T(1) is a disjoint union of uncountably many components on which the right translations act transitively". Later on page the authors wrote a paragraph; "**4.2. Weil-Petersson metric on the universal Teichmüller space.** In this section we consider T(1) as a Hilbert Manifold. The Weil-Petersson metric on T(1) is a Hermitian metric defined by the Hilbert space inner product on tangent spaces, which are identified with the Hilbert space $H^{-1,1}\left(\mathbb{D}^*\right)$ by right translations (see Section 3.3). Thus the Weil-Petersson metric is a right-invariant metric on T(1) defined at the origin of T(1) by*

$$\langle\mu,\nu\rangle_{W.P.}:=\iint\limits_{\mathbb{D}}\mu\overline{\nu}\left(1-|z|^2\right)^2dz^2,\ \mu,\nu\in H^{-1,1}\left(\mathbb{D}^*\right)=T_0T(1).\qquad(4.2)$$

To every $\mu \in H^{-}1,1\left(\mathbb{D}^*\right)$ there corresponds a vector field $\frac{\partial}{\partial\varepsilon_\mu}$, over V_0 given by (2.5)-(2.7). We set for every $\kappa \in V_0$,

$$g_{\mu,\overline{\nu}}(\kappa) := \left\langle \frac{\partial}{\partial\varepsilon_\mu}, \frac{\partial}{\partial\varepsilon_\nu} \right\rangle_{W.P.} = \iint\limits_{\mathbb{D}} P\left(R\left(\mu,\kappa\right)\right) \overline{P\left(R\left(\mu,\kappa\right)\right)} dz^2. \tag{4.3}$$

This formula explicitly defines the Weil-Petersson metric on the coordinate chart V_0." In [20] is claimed that that the inner product on $A_2(\mathbb{D})$ given by

$$\left\|\psi(\overline{z})\right\|^2 := \frac{1}{2\pi\sqrt{-1}} \iint\limits_{\mathbb{D}} \left(1-|z|^2\right)^2 \left|\psi(\overline{z})\right|^2 dz \wedge \overline{dz} = \sum_{n>1} \frac{|a_{n-2}|^2}{n(n^2-1)} \tag{38}$$

induces the Weil-Petersson metric. This is clearly the Hilbert space of the boundary values of holomorphic functions with finite $-3/2$ Sobolev norm. The Weil-Petersson metric on the $\mathbf{T}^\infty$ should be induced by the norm of the Hilbert space of the boundary values of holomorphic functions with finite $3/2$ Sobolev norm. See [16], [22] and [21].

4 $3/2$ Hilbert Complex Analytic Structure on $\mathbf{T}^\infty$

4.1 Special Properties of Two Operators

Definition 27 *Let $h(z) \in \mathbf{L}^p\left(\mathbb{C}\right)$, define*

$$P\left(h((\varsigma)\right) := -\frac{1}{\pi} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} h(z) \left(\frac{1}{z-\varsigma} - \frac{1}{z}\right) dxdy, \tag{39}$$

where $z = x+iy$. See [1] chapter V, part A.

Definition 28 *Let $h(z) \in C_0^2\left(\mathbb{C}\right)$, define*

$$T\left(h\left(\varsigma\right)\right) := \lim_{\varepsilon \to 0} \left(-\frac{1}{\pi} \int\limits_{|z-\varsigma|^2>\varepsilon} h(z) \left(\frac{1}{z-\varsigma}\right)^2 dxdy \right), \tag{40}$$

where $z = x+iy$. See [1] chapter V, part A.

The following Lemmas were proved in [1] Chapter V, part A:

Lemma 29 *Let $h(z) \in \mathbf{L}^p\left(\mathbb{C}\right)$, $p > 2$. The function Ph is continuous and satisfies Hölder condition, i.e.*

$$\left|Ph\left(\varsigma_1\right) - Ph\left(\varsigma_2\right)\right| \leq K_p \left\|h\right\|_p \left|\varsigma_1 - \varsigma_2\right|^{1-\frac{2}{p}}.$$

Lemma 30 *For $h \in \mathbf{L}^p(\mathbb{C})$, $p > 2$, the following relations hold*

$$(Ph)_{\overline{z}} = h \ \& \ (Ph)_z = Th. \tag{41}$$

Theorem 31 *For $h \in \mathbf{L}^p$, $p > 1$ on $\mathbb{C}$, we have*

$$\|Th\|_p \leq C_p \|h\|_p \tag{42}$$

and

$$\lim_{p\to 2} C_p = 1. \tag{43}$$

4.2 Basis of $\ker\left(\overline{\partial}_{\mu_\psi}\right)$ for $\mu_\psi \in \mathbb{H}_1^{-1,1}(\mathbb{D})$

Theorem 32 was proved in [19]. The ideas of the proof of Theorem 32 are the same as in [19]. Some of the technical steps are simplified in the present proof.

Theorem 32 *Let $\psi(\overline{z})$ be an antiholomorphic function in the unit disk such that*

$$\psi(\overline{z})\,|_{S^1} \in \mathbf{H}^{-3/2}_{S^1,h} \ \ \& \ \ \|\psi(\overline{z})\,|_{S^1}\|^{-3/2}_{S^1,h} \leq c < 1.$$

Let $\mu_\psi(z) = \left(1-|z|^2\right)^2 \psi(\overline{z})\left(\overline{dz}\otimes\frac{d}{dz}\right) \in \mathbb{H}_1^{-1,1}(\mathbb{D})$ be the Beltrami differential defined associated with $\mu_\psi(z) = \left(1-|z|^2\right)^2 \psi(\overline{z})$. We assumed that

$$\|\mu_\psi(z)\|_{\mathbf{L}_\infty} \leq k < 1.$$

Let choose $p > 2$ such that so that $kC_p < 1$, where C_p is defined by Theorem 31, Let T be defined by (40) and $\nu^{(n)}(z)$ be:

$$\nu^{(n)}(z) := \sum_{m=0}^{\infty} T_m^n\left(\mu_\psi(z)\right), \tag{44}$$

where $T_0^n(\mu_\psi) = z^{n-1}$, $T_1^n(\mu_\psi) = T\left(\mu_\psi(z)z^{n-1}\right)$, $T_m^n(\mu_\psi) = T\left(\mu_\psi T_{m-1}^n(\mu_\psi)\right)$ for $m > 0$. Define

$$w^{(n)}(z) = z^n + nP\left(\left(\mu_\psi\left(\nu^{(n)}(z) + z^{n-1}\right)\right)\right). \tag{45}$$

Then for any integer $n > 0$ we have that $w^{(n)}(z)$ is the unique solution of the following problem:

$$\left\{\begin{array}{l} \left(\overline{\partial} - \mu_\psi\partial\right)\left(w^{(n)}(z)\right) = 0, \\ \frac{\partial}{\partial z}\left(w^{(n)}(z)\right) - nz^{n-1} = n\nu^{(n)}(z) \in \mathbf{L}^p(\mathbb{C}), \\ \int\limits_0^{2\pi} w^{(n)}\left(e^{i\theta}\right) d\theta = 0 \end{array}\right. . \tag{46}$$

Proof: The proof of Theorem 32 follows the argument used in Chapter **V** in [1].

Lemma 33 *$\nu^{(n)}(z)$ are well defined functions on $\mathbb{C}$ and $\nu^{(n)}(z) \in \mathbf{L}^p(\mathbb{C})$.*

Proof: The linear operator $h \to T(\mu_\psi(z)h)$ on $\mathbf{L}^p(\mathbb{C})$ has a norm $\leq kC_p < 1$. Therefore the series

$$\nu^{(n)}(z) = \sum_{m=0}^{\infty} T_m^n(\mu_\psi(z)) \tag{47}$$

is converging in $\mathbf{L}^p(\mathbb{C})$. From here we derive that $\nu^{(n)}(z) \in \mathbf{L}^p(\mathbb{C})$. Lemma 33 is proved. ■

Lemma 34 *We have*

$$\nu^{(n)}(z) \in \mathbf{L}^2(\mathbb{C}) \ \ \& \ T\left(\mu_\psi(z)\left(\nu^{(n)}(z)\right)\right) \in \mathbf{L}^2(\mathbb{C}). \tag{48}$$

Proof: According to Theorem 15, the function $\mu_\psi(z)$ is bounded in the unit disk. We extended $\mu_\psi(z)$ outside the unit disk to be zero. Thus $n\mu_\psi(z)z^{n-1}$ is a bounded function on $\mathbb{C}$ and zero outside the unit disk. Thus $\mu_\psi(z) \in \mathbf{L}_0^2(\mathbb{C})$. Ahlfors proved that the operator T defined by (40) defines an isometry of $\mathbf{L}_0^2(\mathbb{C})$. See Lemma **2** in Chapter **V,** part **A** in [1]. The definition of $\nu^{(n)}(z)$ and the above property of T implies that $\nu^{(n)}(z) \in \mathbf{L}^2(\mathbb{C})$. We have

$$\|\mu_\psi(z)\|^2_{\mathbf{L}^2(\mathbb{D})} = \frac{\sqrt{-1}}{2\pi}\int\limits_{\mathbb{D}} \left(\left(1-|z|^2\right)^2 |\psi(z)|\right)^2 dz \wedge \overline{dz}.$$

Applying Schwarz inequality and $\|\mu_\psi(z)\|_{\mathbf{L}^\infty(\mathbb{D})} \leq k < 1$ we get that

$$\|\mu_\psi(z)\|^2_{\mathbf{L}^2(\mathbb{D})} \leq \left(\frac{\sqrt{-1}}{2\pi}\int\limits_{\mathbb{D}} \left(1-|z|^2\right)^2 |\psi(z)|\, dz \wedge \overline{dz}\right)^2 < 2. \tag{49}$$

Combining Theorem 31 with (40) and (49) we get (48). ■

Lemma 35 *Let $w^{(n)}(z)$ be defined by (45). Then **1)** $w^{(n)}(z)$ satisfies in $\mathbb{C}$ the equation:*

$$\overline{\partial} w^{(n)}(z) = n\mu_\psi(z)\left(\nu^{(n)}(z) + z^{n-1}\right), \tag{50}$$

***2)** $w^{(n)} - z^n \in \mathbf{L}^p(\mathbb{C})$ where $p \geq 2$ was already fixed, and **3)** $\overline{\partial} w^{(n)}(z) \in \mathbf{L}^2(\mathbb{C})$ and $w^{(n)}(z) - z^n \in \mathbf{L}^2(\mathbb{C})$.*

Proof of part 1: The definitions of $w^{(n)} = z^n + nP\left(\mu_\psi\left(\nu^{(n)}(z)\right) + z^{n-1}\right)$ and of $\nu^{(n)}(z)$ given by (44) imply

$$w^{(n)}(z) - z^n = nP\left(\mu_\psi\left(\nu^{(n)}(z) + z^{n-1}\right)\right).$$

Lemma 30 and $\overline{\partial} \circ P = id$ imply that

$$\overline{\partial} w^{(n)}(z) = \overline{\partial}\left(z^n + nP\left(\mu_\psi\left(\nu^{(n)}(z) + z^{n-1}\right)\right)\right) = n\mu_\psi(z)\left(\nu^{(n)}(z) + z^{n-1}\right).$$

Part **1** of Lemma 35 is proved. ■

Proof of part 2: Ahlfors proved in Part **V, B** in [1] that if

$$\mu_\psi\left(\nu^{(n)}(z)\right)\in\mathbf{L}^p(\mathbb{C}),$$

then

$$P\left(\mu_\psi\left(\nu^{(n)}(z)+z^{n-1}\right)\right)\in\mathbf{L}^p(\mathbb{C})$$

Part **2** of Lemma 35 is proved. ■

Proof of part 3: Part **3** follows directly from (50) and (48). ■

Lemma 36 *We have*

$$\partial w^{(n)}(z)-nz^{n-1}=n\nu^{(n)}(z)\in\mathbf{L}^2(\mathbb{C}). \tag{51}$$

Proof: We assumed that $w^{(n)}(z)$ is defined by (45). So

$$\partial w^{(n)}(z)-nz^{n-1}=n\partial\left(P\left(\mu_\psi\left(\nu^{(n)}(z)+z^{n-1}\right)\right)\right).$$

Thus $\partial P=T$ implies

$$\partial w^{(n)}-nz^{n-1}=n\partial\left(P\left(\mu_\psi\left(\nu^{(n)}(z)+z^{n-1}\right)\right)\right)=nT\left(\mu_\psi\left(\nu^{(n)}(z)+z^{n-1}\right)\right).$$

The definition of the function $\nu^{(n)}(z)$ implies that

$$T\left(\mu_\psi\left(\nu^{(n)}(z)+z^{n-1}\right)\right)=\nu^{(n)}(z).$$

Thus

$$\partial w^{(n)}-nz^{n-1}=n\nu^{(n)}(z). \tag{52}$$

Since μ_ψ has a compact support and thus $\mu_\psi\left(\nu^{(n)}(z)\right)$, then

$$\mu_\psi\left(\nu^{(n)}(z)\right)\in\mathbf{L}^2(\mathbb{C}). \tag{53}$$

Lemma 36 follows from (52), (53), the definition (44) of $\nu^{(n)}(z)$ and Theorem 31. ■

Lemma 37 *Let $w^{(n)}(z)$ be defined by (45). Let $\Phi_\infty^{(n)}(z):=w^{(n)}(z)\mid_{\mathbb{D}^*}$. Then $\Phi_\infty^{(n)}(z)$ is a holomorphic function given by*

$$\Phi_\infty^{(n)}(z)=z^n+\sum_{i=1}^{\infty}b_iz^{-i}. \tag{54}$$

Proof: Since $\mu_\psi=0$ in $\mathbb{C}-\mathbb{D}=\mathbb{D}^*$ Lemma 38 implies

$$\overline{\partial}w^{(n)}(z)\mid_{\mathbb{D}^*}=n\overline{\partial}P\left[\mu_\psi\left(\nu^{(n)}+z^{n-1}\right)\right]\mid_{\mathbb{D}^*}=$$

$$\mu_\psi(z)\left(\nu^{(n)}+z^{n-1}\right)|_{\mathbb{D}^*}=0.$$

Thus

$$\Phi_\infty^{(n)}(z):=w^{(n)}(z)\,|_{\mathbb{D}^*}=z^n+G(z)+\sum_{i=0}^{\infty}b_iz^{-i},$$

where $G(z)$ is an analytic function defined in $\mathbb{C}$. Since by [1]

$$\partial_z w^{(n)}(z)-nz^{n-1}=n\nu^{(n)}(z)\in\mathbf{L}^p(\mathbb{C})$$

we have $G(z)=A_0$ is a constant. So we we can choose $A_0=0$. Thus $\Phi_\infty^{(n)}(z)$ satisfies

$$\Phi_\infty^{(n)}(z)=z^n+\sum_{i=0}^{\infty}b_iz^{-i}.$$

Thus $\int\limits_{S^1} w^{(n)}(z)\frac{dz}{z}=0$. Cauchy's theorem implies $\int\limits_{S^1} w^{(n)}(z)\frac{dz}{z}=A_0=0$. So $w^{(n)}(z)$, satisfies all the three conditions in (46). Lemma 37 is proved. ■

Corollary 38 *$w^n(z)$ satisfies the Beltrami equation, i.e.*

$$\overline{\partial}w^{(n)}(z)-\mu_\psi(z)\partial w^{(n)}(z)=0.$$

Proof: According to (50) we have $\overline{\partial}w^{(n)}(z)=n\mu_\psi(z)\left(\nu^{(n)}(z)\right)$. We defined

$$w^{(n)}(z):=z^n+nP\left(\mu_\psi(z)\left(\nu^{(n)}(z)+z^{n-1}\right)\right)$$

Lemma 36 implies

$$\partial w^{(n)}(z)=nz^{n-1}+n\nu^{(n)}(z). \tag{55}$$

Substituting (55) in (50) we get $\overline{\partial}w^{(n)}(z)-\mu_\psi(z)\partial w^{(n)}(z)=0$. Corollary 38 is proved. ■

Next we need to prove uniqueness of the solutions of the Beltrami equation. If the Beltrami equation has two solutions $w_i^{(n)}(z)$ that satisfy the three conditions in (46) then the difference $w_2^{(n)}(z)-w_1^{(n)}(z)=W^{(n)}(z)$ will be also a solution of the Beltrami equation. Therefore $W^{(n)}|_{\mathbb{D}^*}$ will be a holomorphic function in $\mathbf{L}^p$. From here we deduce that $W^{(n)}(\infty)=0$ and thus we found a bounded solution $W^{(n)}(z)$ of the Beltrami equation in $\mathbb{P}^1(\mathbb{C})$. But the Beltrami equation is an elliptic. So by maximum principle of elliptic equations it follows that $W^{(n)}(z)$ is a constant and since $W^{(n)}(\infty)=0$ we get that it is zero. Theorem 32 is proved. ■

Corollary 39 *The functions $w^{(n)}(z)$ are continuous functions on $\mathbb{C}$.*

Proof: Corollary 39 follows from Theorem 32 directly. ■

4.3 The Embedding of $\mathbb{H}_1^{-1,1}(\mathbb{D})$ into the space of Hilbert-Schmidt Operators on $\mathbb{HS}(\mathbf{H}^+, \mathbf{H}^-)$.

We recall that for $\mathbf{H}_1$ and $\mathbf{H}_2$ Hilbert spaces, an operator $T : \mathbf{H}_1 \to \mathbf{H}_2$ is called Hilbert-Schmidt if for any orthogonal basis $\{e_i\}_{i\in I}$ of $\mathbf{H}_1$ we have

$$\sum_{i\in I} \|Te_i\|^2 < \infty.$$

If this is the case for one basis of $\mathbf{H}_1$ it is also the case for every basis of $\mathbf{H}_1$.

Theorem 40 *Let $\mu_\psi \in \mathbb{H}^{-1,1}(\mathbb{D})$ be a Beltrami differential where $\mathbb{H}^{-1,1}(\mathbb{D})$ is defined by (3). Let*

$$\|\mu_\psi\|_{\mathbf{L}^\infty(\mathbb{D})} \leq k < 1.$$

Then

$$\mathbf{W}_{\mu_\psi} := \ker \overline{\partial}_{\mu_\psi} := \left\{ f(z)\,|_{S^1} \in \mathbf{L}^2\left(S^1\right) \left| \overline{\partial}_{\mu_\psi}\left(f(z)\right) = 0 \right. \right\}$$

is a closed Hilbert subspace in $\mathbf{L}^2\left(S^1\right)$ and the projection operator $\mathbf{pr}^+_{\mu_\psi} : \mathbf{W}_{\mu_\psi} \to \mathbf{H}^+$ is an isomorphism. See [19].

Proof: First we will prove that

Lemma 41 $\mathbf{W}_{\mu_\psi}$ *is a closed Hilbert space in $\mathbf{L}^2\left(S^1\right)$.*

Proof: In [1] it is proved that $\left\{ \left(w^{(1)}(z)\right)^n |_{S^1} \right\}$ form an orthonormal basis on $\mathbf{W}_{\mu_\psi}$ and any function $f(z)$ on $\mathbb{C}$ that satisfies the Beltrami equation and has a finite $\mathbf{L}^2$ norm when restricted on S^1 can be expressed as follows:

$$f(z)\,|_{S^1} := \sum_{m\geq 0} a_n \left(w^{(1)}(z)\Big|_{s^1} \right)^n \ \& \ \sum_{m\geq 0} |a_n|^2 < \infty.$$

So $\mathbf{W}_{\mu_\psi} := \ker \overline{\partial}_{\mu_\psi}\big|_{s^1}$ is a closed Hilbert space in $\mathbf{L}^2\left(S^1\right)$. ■

Let $w^{(n)}(z)$ be defined by (44) and (45). We know by Lemma 35 that $w^{(n)}(z) - z^n \in \mathbf{L}^2(\mathbb{C})$. Thus $w^{(n)}(z)\big|_{s^1} \in \mathbf{L}^2(S^1)$. Theorem 32 implies that

$$\{ w^{(n)}(z)\Big|_{s^1},\ n = 1, ...\}$$

is a basis of the Hilbert space $\ker \overline{\partial}_{\mu_\psi} = \mathbf{W}_{\mu_\psi}$. From the definition of $w^{(n)}(z)\big|_{s^1}$ and Lemma 37 it follows that $\mathbf{pr}^+_{\mu_\psi}\left(w^{(n)}(z)\big|_{s^1} \right) = z^n|_{s^1}$ for $n = 1, ...$ is an isomorphism between Hilbert spaces. This proves Theorem 40. ■

Theorem 42 *Let $\mu_\psi \in \mathbb{H}^{-1,1}(\mathbb{D})$ be a Beltrami differential where $\mathbb{H}^{-1,1}(\mathbb{D})$ is defined by (3) and*

$$\|\mu_\psi\|_{\mathbf{L}^\infty(\mathbb{D})} \leq k < 1.$$

The projection operator $\mathbf{pr}^-_{\mu_\psi} : \mathbf{W}_{\mu_\psi} \to \mathbf{H}^+$ is a Hilbert-Schmidt operator.

Theorem 42 was proved in [19]. The proof presented in this paper is technically much simpler.

Proof: The proof that $\mathbf{pr}^{-}_{\mu_\psi} : \mathbf{W}_{\mu_\psi} \to \mathbf{H}^{-}$ is a Hilbert-Schmidt operator is based on the following observation. According to Theorem 40 $\mathbf{pr}^{+}_{\mu_\psi} : \mathbf{W}_{\mu_\psi} \to \mathbf{H}^{+}$ is an isomorphism. Therefore if we prove that

$$\mathbf{pr}^{-}_{\mu_\psi}\left(\mathbf{pr}^{+}_{\mu_\psi}\right)^{-1} : \mathbf{H}^{+} \to \mathbf{H}^{-} \tag{56}$$

is Hilbert-Schmidt operator then it will follow that $\mathbf{pr}^{-}_{\mu_\psi} : \mathbf{W}_{\mu_\psi} \to \mathbf{H}^{+}$ is Hilbert-Schmidt too. The proof of this fact is based on several Lemmas.

Lemma 43 *We have*

$$\mathbf{pr}^{-}_{\mu_\psi}\left(w^{(n)}(z)\Big|_{s^1}\right) =$$
$$n\ P\left[\mu_\psi\left(\left(\sum_{m=1}^{\infty} T^{n}_{m}\left(\mu_\psi\right) + z^{n-1}\right)\right)\right]\Bigg|_{S^1} = \sum_{n>0} a_n e^{-in\theta}. \tag{57}$$

Proof: According to Lemma 37 and Theorem 32 the restriction of $w^{(n)}(z)$ on $\mathbb{D}^*$ is a complex analytic function such that it is in $\mathbf{L}^{p}(\mathbb{C})$. So (54) implies that

$$w^{(n)}(z)\Big|_{\mathbb{D}^*} = z^n + \sum_{j>1} a_j z^{-j}. \tag{58}$$

The space H_{-} defined by (13) is spanned by z^{-j} for $j > 0$. On the other hand Theorem 32 implies that $w^{(n)}(z)\big|_{s^1}$ are restriction of holomorphic $\mathbf{L}^2$ functions from $\mathbb{D}^*$ on S^1 and they form a basis of $\mathbf{W}_{\mu_\psi}$. Thus (58) implies that

$$\mathbf{pr}^{-}_{\mu_\psi}\left(w^{(n)}(z)\Big|_{s^1}\right) = w^{(n)}(z)\Big|_{s^1} - z^n\big|_{s^1} = \sum_{n>0} a_n e^{-in\theta}.$$

Lemma 43 implies that

$$\mathbf{pr}^{-}_{\mu_\psi}\left(w^{(n)}(z)\Big|_{s^1}\right) = n\ P\left[\mu_\psi\left(\sum_{m=1}^{\infty}(T^{n}_{m}\left(\mu_\psi\right) + z^{n-1}\right)\right]\Bigg|_{S^1} =$$
$$\sum_{n>0} a_n e^{-in\theta} \in \mathbf{L}^2(S^1). \tag{59}$$

Thus (57) is proved. Lemma 43 follows directly. ■

Lemma 44 *Let* $\left\langle \mathbf{pr}^{-}_{\mu_\psi}\left(\mathbf{pr}^{+}_{\mu_\psi}\right)^{-1}\left(z^n\big|_{s^1}\right), z^{-k}\big|_{s^1}\right\rangle$ *denote the scalar product in the Hilbert space* $\mathbf{L}^2(S^1)$*. Then we have*

$$\sum_{n=1}^{\infty}\left\|\mathbf{pr}^{-}_{\mu_\psi}\left(\mathbf{pr}^{+}_{\mu_\psi}\right)^{-1}\left(z^n\big|_{s^1}\right)\right\|^2_{\mathbf{L}^2(S^1)} =$$
$$\sum_{n,k=1}^{\infty}\left|\left\langle \mathbf{pr}^{-}_{\mu_\psi}\left(\mathbf{pr}^{+}_{\mu_\psi}\right)^{-1}\left(z^n\big|_{s^1}\right), z^{-k}\big|_{s^1}\right\rangle\right|^2 < \infty. \tag{60}$$

Proof: We are going to use the convention that $z = r\exp(i\theta)$. An elementary application of Stokes' theorem and (57) gives that

$$\left\langle \mathbf{pr}^{-}_{\mu_\psi}\left(\mathbf{pr}^{+}_{\mu_\psi}\right)^{-1}\left(z^n|_{s^1}\right), z^{-k}|_{s^1}\right\rangle =$$

$$\frac{1}{2\pi}\int_0^{2\pi}\mathbf{pr}^{-}_{\mu_\psi}\left(\mathbf{pr}^{+}_{\mu_\psi}\right)^{-1}\left(z^n|_{s^1}\right)\overline{\exp(-ik\theta)}\,d\theta =$$

$$\frac{1}{2\pi i}\int_{S^1}\mathbf{pr}^{-}_{\mu_\psi}\left(\mathbf{pr}^{+}_{\mu_\psi}\right)^{-1}\left(z^n|_{s^1}\right) z^k\frac{dz}{z}\bigg|_{S^1} =$$

$$\frac{1}{2\pi i}\int_{\mathbb{D}}\overline{\partial}\left(\mathbf{pr}^{-}_{\mu_\psi}\left(\mathbf{pr}^{+}_{\mu_\psi}\right)^{-1}(z^n)\right)\wedge(z^{k-1}dz) =$$

$$\frac{1}{2\pi i}\int_{\mathbb{D}}\overline{\partial}P\left[n\mu_\psi\left(\sum_{m=1}^{\infty}(T^n_m(\mu_\psi)+z^{n-1}\right)\right]\overline{dz}\wedge(z^{k-1}dz). \tag{61}$$

But, from the construction of

$$w^{(n)} = z^n + nP\left[\mu_\psi\left(\sum_{m=1}^{\infty}(T^n_m(\mu_\psi)+z^{n-1}\right)\right]$$

$\overline{\partial P} = id$, and the definition of $\nu^{(n)}(z)$ we have that

$$\overline{\partial}\left(P\left[\mu_\psi\left(\sum_{m=1}^{\infty}(T^n_m(\mu_\psi)+z^{n-1}\right)\right]\right) = \nu^{(n)}(z). \tag{62}$$

So substituting (62) into (61) we get that

$$\left\langle \mathbf{pr}^{-}_{\mu_\psi}\left(\mathbf{pr}^{+}_{\mu_\psi}\right)^{-1}\left(z^n|_{s^1}\right), z^{-k}|_{s^1}\right\rangle =$$

$$\frac{1}{2\pi i}\int_{\mathbb{D}}\left(n\mu_\psi\left(\sum_{m=1}^{\infty}T^n_m(\mu_\psi)+z^{n-1}\right)\right)z^{k-1}dz\wedge\overline{dz}. \tag{63}$$

Cauchy-Schwarz (for the inner-product of $\mathbf{L}^2(\mathbb{D})$), applied to the right hand side of equation (63) gives:

$$\left|\left\langle \mathbf{pr}^{-}_{\mu_\psi}\left(\mathbf{pr}^{+}_{\mu_\psi}\right)^{-1}\left(z^n|_{s^1}\right), z^{-k}|_{s^1}\right\rangle\right|^2 =$$

$$\left|\frac{1}{2\pi i}\int_{\mathbb{D}}n\mu_\psi\left(\sum_{m=1}^{\infty}T^n_m(\mu_\psi)+z^{n-1}\right)z^{k-1}dz\wedge\overline{dz}\right|^2 \leq$$

$$\left(\frac{C}{2\pi i}\int\limits_{\mathbb{D}} n^2\left|\sum_{m=1}^{\infty} T_m^n\left(\mu_\psi\right)+\mu_\psi z^{n-1}\right|^2 dz\wedge\overline{dz}\right)\left(\frac{1}{2\pi i}\int\limits_{\mathbb{D}}\left(\mu_\psi\right)^2|z|^{2k-2}\,dz\wedge\overline{dz}\right)\leq$$

$$n^2\left(\left\|\sum_{m=1}^{\infty} T_m^n\left(\mu_\psi\right)\right\|^2_{\mathbf{L}^2(\mathbb{D})}+\frac{1}{2\pi i}\int\limits_{\mathbb{D}}\left(\mu_\psi\right)^2|z|^{2n-2}\;d\bar{z}\wedge dz\right)\times$$

$$\frac{1}{2\pi i}\int\limits_{\mathbb{D}}\left(\mu_\psi\right)^2|z|^{2k-2}\,dz\wedge\overline{dz}. \tag{64}$$

We are going to estimate each of the norms of equation (64).

Proposition 45 *The following inequality holds for $n>2$:*

$$\frac{1}{2\pi i}\int\limits_{\mathbb{D}}\left(\mu_\psi\right)^2|z|^{2n-2}\,d\bar{z}\wedge dz\leq\frac{C}{n\left(n+1\right)\left(n+2\right)\left(n+3\right)}. \tag{65}$$

Proof of (65)**:** We have

$$\frac{1}{2\pi i}\int\limits_{\mathbb{D}}\left(\mu_\psi\right)^2|z|^{2n-2}\,d\bar{z}\wedge dz=$$

$$\frac{1}{2\pi}\int\limits_0^{2\pi}\int\limits_0^1(1-r^2)^4\left|\psi\left(\overline{z}\right)\right|^2r^{2n-1}\;dr\;d\theta=\int\limits_0^1(1-r^2)^4\left|\psi\left(\overline{z}\right)\right|^2r^{2n-1}\;dr.$$

By Theorem (16) we have $\left|\psi\left(|z|\right)\right|<C\left(1-|z|\right)^{-\alpha}$ for $0<\alpha<1$. So

$$\frac{1}{2\pi i}\int\limits_{\mathbb{D}}\left(\mu_\psi\right)^2|z|^{2n-2}\,d\bar{z}\wedge dz\leq\int\limits_0^1\left((1-r^2)^2\left|\psi\left(\overline{z}\right)\right|\right)^2(1-r^2)^2r^{2n-1}dr\leq$$

$$c\int\limits_0^1\left((1-r^2)^{2(1-\alpha)}\right)(1-r^2)^2r^{2n-1}dr.$$

Direct computations by integrations by parts and using that $0<\alpha<1$ and thus $-1\leq 1-2\alpha$ we get

$$\int\limits_0^1(1-r^2)^{4-2\alpha}r^{2n-1}dr=\frac{1}{2n}\int\limits_0^1(1-r^2)^{4-2\alpha}dr^{2n}=$$

$$-\frac{1}{2n}\int\limits_0^1 r^{2n}d\left((1-r^2)^{4-2\alpha}\right)=-\frac{(4-2\alpha)}{2n}\int\limits_0^1 r^{2n+1}(1-r^2)^{3-2\alpha}dr$$

$$\frac{(4-2\alpha)}{2n}\int_0^1 (1-r^2)^{3-2\alpha} dr^{2n+2} = -\frac{(4-2\alpha)}{2n\,(2n+2)}\int_0^1 (1-r^2)^{3-2\alpha} dr^{2n+2} =$$

$$\frac{(4-2\alpha)}{2n\,(2n+2)}\int_0^1 r^{2n+2} d(1-r^2)^{3-2\alpha} = \frac{(4-2\alpha)}{2n\,(2n+2)}\int_0^1 r^{2n+3}(1-r^2)^{2-2\alpha} dr.$$

Integrating twice by parts the expression

$$\frac{(4-2\alpha)}{2n\,(2n+2)}\int_0^1 r^{2n+3}(1-r^2)^{2-2\alpha} dr$$

we get

$$\frac{1}{2\pi i}\int_{\mathbb{D}} |\mu_\psi z|^{2n-2}\, d\bar{z}\wedge dz =$$

$$\int_0^1 (1-r^2)^{4-2\alpha} r^{2n-1} dr \leq \frac{C_1}{n\,(n+1)\,(n+2)\,(n+3)}. \tag{66}$$

(65) is proved. ■

Proposition 46 *Let* $\psi(\overline{z}) = \sum\limits_{p=2} a_p \overline{z}^{p-1}$. *Then*

$$\left\| \sum_{m=0}^{\infty} T_m^n\left(\mu_\psi\right) \right\|^2_{\mathbf{L}^2(\mathbb{D})} =$$

$$\left\| \sum_{m=1}^{\infty} (T_m^n\left(\mu_\psi\right) + \mu_\psi z^{n-1}) \right\|^2_{\mathbf{L}^2(\mathbb{D})} \leq \left(\frac{c_0}{1-c_0} + \left\|\mu_\psi z^{n-1}\right\|^2_{\mathbf{L}^2(\mathbb{D})} \right) \leq$$

$$C_0 \sum_{p=2}^{\infty} \frac{1}{n\,(n+1)\,(n+2)(n+3)} < \infty. \tag{67}$$

Proof: According to [1] the Hilbert transform T is an isometry of $\mathbf{L}^2(\mathbb{C})$. By the definition of the operators $T_m^n(\mu_\psi) = T\left(\mu_\psi T_{m-1}^n(\mu_\psi)\right)$and $T_0^n = \mu_\psi z^{n-1}$ and and the above results in [1] imply that we have

$$\left\|T_m^n\right\|^2 \leq c_0 \left\|T_{m-1}^n\right\|^2 \leq ... \leq c_0^n \left\|\mu_\psi z^{n-1}\right\|^2_{\mathbf{L}^2(\mathbb{D})},$$

where $c_0 \overset{def}{=} \|\mu_\psi\|_{\mathbf{L}^\infty(\mathbb{D})} < 1$. Thus we have

$$\left\| \sum_{m=1}^{\infty} (T_m^n\left(\mu_\psi\right) \right\|^2_{\mathbf{L}^2(\mathbb{D})} \leq \frac{c_0}{1-c_0}. \tag{68}$$

Next we need to estimate

$$\left\|\left(\mu_\psi z^{n-1}\right)\right\|^2_{\mathbf{L}^2(\mathbb{D})}=\frac{1}{2\pi\sqrt{-1}}\int\limits_{\mathbb{D}}\left|\mu_\psi z^{n-1}\right|^2 dz\wedge\overline{dz}=$$

$$\frac{1}{2\pi\sqrt{-1}}\int\limits_{\mathbb{D}}\mu_\psi^2\left|z\right|^{2n-2}dz\wedge\overline{dz}.$$

We proved that μ_ψ is a bounded function on $\mathbb{C}$. (65) implies

$$\left\|\left(\mu_\psi z^{n-1}\right)\right\|^2_{\mathbf{L}^2(\mathbb{D})}\leq\frac{C}{n\left(n+1\right)\left(n+2\right)\left(n+3\right)}. \tag{69}$$

$$\left\|\sum_{m=0}^{\infty}\left(T_m^n\left(\mu_\psi\right)\right\|^2_{\mathbf{L}^2(\mathbb{D})}\leq\frac{C_1}{n\left(n+1\right)\left(n+2\right)\left(n+3\right)}$$

Proposition 46 is proved. ■

To conclude the proof of Lemma 44 we will use the two estimates (65) and (67) above for the norms that appear on the right hand side of equations (63) and (64) to get

$$\sum_{n=1}^{\infty}\left\|\mathbf{pr}^-_{\mu_\psi}\left(\mathbf{pr}^+_{\mu_\psi}\right)^{-1}\left(z^n\right)\right\|^2_{\mathbf{L}^2(S^1)}=\sum_{n,k=1}^{\infty}\left|\left\langle\nu^{(n)}(z)\Big|_{s^1},\,z^{-k}\Big|_{s^1}\right\rangle\right|^2<$$

$$\sum_{k=1,n=1,m=1}^{\infty}\left(\left\|\mu_\psi z^{k-1}\right\|^2_{\mathbf{L}^2(\mathbb{D})}\right)\times\left(n^2\left\|\left(T_m^n\left(\mu_\psi\right)\right)\right\|^2_{\mathbf{L}^2(\mathbb{D})}\right)=$$

$$C\left(\sum_{k=1}^{\infty}\frac{1}{k\left(k+1\right)\left(k+2\right)(k+3)}\right)\left(\sum_{n=1}^{\infty}\frac{n^2}{n\left(n+1\right)\left(n+2\right)\left(n+3\right)}\right)<\infty.$$

Lemma 44 is proved. ■ Lemma 44 implies Theorem 42. ■

4.4 Differential Geometry of the Segel Wilson Grassmannian

Definition 47 ***1.*** *The Segal-Wilson Grassmannian $\mathbb{G}\mathrm{r}_\infty$ is defined as the set of all closed subspaces $\mathbf{W}$ of $\mathbf{L}^2(S^1)=\mathbf{H}^+\oplus\mathbf{H}^-$ such that the projection*

$$\mathrm{pr}_+:\mathbf{W}\to\mathbf{H}^+$$

is Fredholm and the projection

$$\mathrm{pr}_-:\mathbf{W}\to\mathbf{H}^-$$

is Hilbert-Schmidt, where $\mathbf{H}$ is the $\mathbf{L}^2$ space of complex functions on S^1. ***2.*** *A linear operator $A\in\mathbb{GL}_{\mathrm{res}}(\mathbf{L}^2(S^1))$ iff the following conditions are satisfied:* ***A.*** *A is an invertible bounded linear operator of $\mathbf{L}^2(S^1)$ onto itself.* ***B.*** *If we write*

A in block matrix form with respect to the decomposition $\mathbf{L}^2(S^1) = \mathbf{H}^+ \oplus \mathbf{H}^-$ as

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{70}$$

then, the operators b and c are Hilbert-Schmidt operators.

In [17] it is proved that $\mathbb{GL}_{\text{res}}(\mathbf{L}^2(S^1))$ acts transitively on the Segal-Wilson Grassmannian so it is a homogeneous space. It has a natural left invariant metric defined in Section 7.8 of [17]. It is enough to construct the metric at the point $\mathbf{H}^+ = \mathbf{H}^+ \oplus \{0\} \in \mathbb{G}r_\infty$. Note that

$$T_{\mathbf{H}^+,\mathbb{G}r_\infty} = \mathbb{HS}(\mathbf{H}^+,\mathbf{H}^-),$$

where $\mathbb{HS}(\mathbf{H}^+, \mathbf{H}^-)$ denotes the space of Hilbert-Schmidt operators from $\mathbf{H}^+$ into $\mathbf{H}^-$. Hence there is a naturally defined scalar product on $T_{H_+,\mathbb{G}r_\infty}$ by ψ and χ in

$$\langle \psi, \chi \rangle \overset{def}{=} Tr(\psi^* \chi), \tag{71}$$

where ψ and $\chi \in T_{H_+,\mathbb{G}r_\infty}$ [1].

We proved in [19] the following Theorem:

Theorem 48 *Let $\{\psi_k\}$ be an orthonormal basis of*

$$T_{\mathbf{H}^+,\mathbb{G}r_\infty} = \mathbb{HS}(\mathbf{H}^+,\mathbf{H}^-) \subset Hom\left(\mathbf{H}^+,\mathbf{H}^-\right).$$

Then,

a. *The left invariant metric defined by (71) is Kähler and the component of the curvature tensor with respect to the orthonormal basis $\{\psi_k\}$ is given by:*

$$R_{i\bar{j},k\bar{l}} = -\delta_{i\bar{j}}\delta_{k\bar{l}} - \delta_{i\bar{l}}\delta_{k\bar{j}} + Tr\left(\psi_i\psi_j^* \wedge \psi_k\psi_l^*\right) + Tr\left(\psi_i\psi_l^* \wedge \psi_k\psi_j^*\right)$$

for $(i,j) \neq (k,l)$ and $R_{i\bar{j},i\bar{j}} = -2\delta_{i\bar{j}} + Tr\left(\psi_i\psi_j^* \wedge \psi_k\psi_l^*\right)$. **b.** *Let ψ be any complex direction in the tangent space $T_{\mathbf{H}^+}\mathbb{G}r_\infty$. Let K_ψ be the Gaussian sectional curvature in any the two-dimensional real space defined by $\operatorname{Re}\psi$ and $\operatorname{Im}\psi$. Then we have $K\psi = -2 + Tr\left(\psi\psi^* \wedge \psi\psi^*\right) < -\frac{3}{2}$.*

4.5 The Embedding of $\mathbb{H}^{-1,1}(\mathbb{D})$ into the Tangent Space at a point of the Segal-Wilson Grassmannian

Theorem 49 *Let $\mu_\psi \in \mathbb{H}^{-1,1}(\mathbb{D})$, where $\mathbb{H}^{-1,1}(\mathbb{D})$ is defined by (3). Let us define the operator $A_{\mu_\psi} : \mathbf{H}^+ \to \mathbf{H}^-$ affiliated with the Beltrami differential $\mu_\psi\left(\overline{dz} \otimes \frac{d}{dz}\right)$ as follows:*

$$A_{\mu_\psi}(f(z)) := \frac{1}{\lambda}\mathbf{pr}^-_{\lambda\mu_\psi}\left(\left(\mathbf{pr}^+_{\lambda\mu_\psi}\right)^{-1}(f(z))\right), \tag{72}$$

[1] Our definition differs from the one used in [17] by a factor of 2

where λ is such a positive number that

$$\|\lambda\mu_\psi\|_{\mathbf{L}^\infty(\mathbb{D})} = \left\|\lambda(1-|z|^2)^2\psi(\overline{z})\right\|_{\mathbf{L}^\infty(\mathbb{D})} \leq k < 1.$$

Then the map

$$\iota : \mu_\psi \left(\overline{dz} \otimes \frac{d}{dz}\right) \to A_{\mu_\psi}, \tag{73}$$

*defines an embedding of the Hilbert space $\mathbb{H}^{-1,1}(\mathbb{D})$ into the Hilbert space $T_{\mathbf{H}^+,\mathbb{G}r_\infty} = \mathbb{HS}\left(\mathbf{H}^+,\mathbf{H}^-\right)$ and **so** $\iota\left(\mathbb{H}^{-1,1}(\mathbb{D})\right)$ is a closed Hilbert subspace in $T_{\mathbf{H}^+,\mathbb{G}r_\infty} = \mathbb{HS}\left(\mathbf{H}^+,\mathbf{H}^-\right)$.*

Proof of part 1: The definition of $\mathbb{H}^{-1,1}(\mathbb{D})$, Theorem 42 and Theorem 40 imply that the map given by (73) defines a map from $\mathbb{H}^{-1,1}(\mathbb{D})$ to the space of Hilbert-Schmidt operators $\mathbb{HS}\left(H_+,H_-\right)$. The proof of Theorem 49 follows from the Lemma bellow:

Lemma 50 *Let μ_{ψ_1} and μ_{ψ_2} be two different elements of $\mathbb{H}_1^{-1,1}(\mathbb{D})$. Then $A_{\mu_{\psi_1}} \neq A_{\mu_{\psi_2}}$ in $\mathbb{HS}\left(\mathbf{H}^+,\mathbf{H}^-\right)$.*

Proof: The definition of the linear operators $A_{\mu_{\psi_i}}$ implies that $\mathbf{W}_{\mu_{\psi_i}}$ are the graphs of the operators $A_{\mu_{\psi_i}}$. If we prove that the graphs $\mathbf{W}_{\mu_{\psi_1}}$ and $\mathbf{W}_{\mu_{\psi_2}}$ of the operators $A_{\mu_{\psi_i}}$ are different Hilbert subspaces in $\mathbf{L}^2(S^1)$ then $A_{\mu_{\psi_1}} \neq A_{\mu_{\psi_2}}$.

Let us consider the unique quasi-conformal maps $\Phi_i(z)$ of $\mathbb{C}\cup\infty := \mathbb{CP}^1$ defined by

$$\left(\overline{\partial} - \mu_{\psi_i}\partial\right)\left(\Phi_i(z)\right) = 0$$

for $i = 1,2$ which satisfy condition (46) of Theorem 32. Let $\Phi_{i,\infty}(z) = \Phi_i(z)\mid_{\mathbb{D}^*}$. We will prove that the assumption $\psi_1(\overline{z}) \neq \psi_2(\overline{z})$ implies that $\Phi_{1,\infty}(\mathbb{D}^*) \neq \Phi_{2,\infty}(\mathbb{D}^*)$. Theorem 49 will follow.

Proposition 51 *Suppose that $\Phi_{1,\infty}(\mathbb{D}^*) \neq \Phi_{2,\infty}(\mathbb{D}^*)$ then $\mathbf{W}_{\mu_{\psi_1}} \neq \mathbf{W}_{\mu_{\psi_2}}$ in $\mathbf{L}^2(S^1)$.*

Proof: If we normalize $\Phi_{i,\infty}(z)\frac{\sqrt{-1}}{2}, \int\limits_{\mathbb{D}^*} \left|\Phi_{i,\infty}(z)\right|^2 dz\wedge\overline{dz} = 1$, then according to [1] the functions

$$\left\{\left(\Phi_{i,\infty}(z)\right)^n,\ \text{for } n = 1,...\right\}$$

are holomorphic in $\Phi_{i,\infty}(\mathbb{D}^*)$ and form an orthogonal bases for $\mathbf{W}_{\mu_{\psi_i}}$. So from this fact we deduce that $\mathbf{W}_{\mu_{\psi_1}} = \mathbf{W}_{\mu_{\psi_2}}$ if an only if $\Phi_{1,\infty}(\mathbb{D}^*) = \Phi_{2,\infty}(\mathbb{D}^*)$. Proposition 51 is proved. ■

Proposition 52 *$\psi_1(\overline{z}) \neq \psi_2(\overline{z})$ implies $\Phi_{1,\infty}(\mathbb{D}^*) \neq \Phi_{2,\infty}(\mathbb{D}^*)$.*

Proof: The proof of Proposition 52 follows from the following

We assumed that the functions $\psi_1(\overline{z})$ is different from $\psi_2(\overline{z})$. So the definition of $\mu_\psi(z) = \left(1-|z|^2\right)^2\psi(\overline{z})$ implies $\mu_{\psi_1} \neq \mu_{\psi_2} \Longleftrightarrow \psi_1 \neq \psi_2$.

Proposition 53 *Suppose that $\psi_1(\overline{z}) \neq \psi_2(\overline{z})$ then for any conformal map A of $\mathbb{D}^*$, $\Phi_{1,\infty}(\mathbb{D}^*) \neq \Phi_{2,\infty}(A(\mathbb{D}^*))$.*

Proof: Suppose that $\psi_1(\overline{z}) \neq \psi_2(\overline{z})$ and for some conformal map A of $\mathbb{D}^*$ we have $\Phi_{1,\infty}(\mathbb{D}^*) = \Phi_{2,\infty}(A(\mathbb{D}^*))$. We will show that this assumption contradicts Theorem 5. One of the basic properties of the Schwarzian states

$$\mathcal{S}\left[\Phi\left(A\left(\frac{1}{z}\right)\right)\Big|_{\mathbb{D}^*}\right] = \mathcal{S}\left[\Phi\left(\frac{1}{z}\right)\Big|_{\mathbb{D}^*}\right]$$

for any conformal A map of $\mathbb{D}^*$. Theorem 5 implies that

$$\psi_1(z) = \mathcal{S}\left[\Phi_{1,\infty}\left(A\left(\frac{1}{z}\right)\right)\Big|_{\mathbb{D}^*}\right] = \psi_2(z) = \mathcal{S}\left[\Phi_{2,\infty}\left(A\left(\frac{1}{z}\right)\right)\Big|_{\mathbb{D}^*}\right].$$

So we get a contradiction. Proposition 53 is proved. ■ Lemma 50 is proved. ■

Lemma 54 *Recall that $\mathbb{H}_1^{-1,1}(\mathbb{D})$ is the subset in $\mathbb{H}^{-1,1}(\mathbb{D})$ which consists of functions $(1-|z|^2)^2\psi(\overline{z})$, where $\psi(\overline{z})$ is antiholomorphic function in the unit disk,*

$$\psi(\overline{z})|_{S^1} \in \mathbf{H}_{S^1,h}^{-3/2}$$

and

$$\left\|(1-|z|^2)^2\psi(\overline{z})\right\|_{\mathbf{L}^\infty(\mathbb{D})} \leq k < 1.$$

Let $\iota_{3/2} : \mathbb{H}_1^{-1,1}(\mathbb{D}) \to \mathbf{H}_{S^1,h}^{3/2}$ be a linear map defined by:

$$\iota_{3/2}\left(\left(1-|z|^2\right)^2\psi(\overline{z})\right) = \Psi(\overline{z})\,|_{S^1},$$

where $\Psi(\overline{z})$ is given by (5). Then $\iota_{3/2}$ is a continuous map and $\iota_{3/2}\left(\mathbb{H}_1^{-1,1}(\mathbb{D})\right)$ is an open set in $\mathbf{H}_{S^1,h}^{3/2}$.

Proof: Let $\Psi_0(\overline{z}) = \iota_{3/2}\left((1-|z|^2\psi(\overline{z})\right) \in \mathbf{H}_{S^1,h}^{3/2}$. We need to show that there exists $\varepsilon, \delta > 0$ such that if

$$\sup_{|z|<1}\left(1-|z|^2\right)^2|\psi(\overline{z}) - \psi_0(\overline{z})| < \delta \tag{74}$$

then

$$\|\Psi(\overline{z}) - \Psi_0(\overline{z})\|^2_{\mathbf{H}_{S^1,h}^{3/2}} < \varepsilon.$$

We proved that

$$\|\Psi(\overline{z}) - \Psi_0(\overline{z})\|^2_{\mathbf{H}_{S^1,h}^{3/2}} = \|\psi(\overline{z}) - \phi_0(\overline{z})\|^2_{\mathbf{H}_{S^1,h}^{-3/2}}.$$

Recall that if

$$\psi(\overline{z}) = \sum_{n=2}^{\infty} a_n \overline{z}^{n-2},$$

then

$$\Psi(\overline{z}) = \sum_{n=2}^{\infty} \frac{a_n}{n\left(n^2-1\right)} \overline{z}^{n-2} = \sum_{n=2}^{\infty} b_n \overline{z}^{n-2}.$$

Then (74) implies that

$$\left\| \left(\psi(\overline{z}) - \psi_0(\overline{z})\right)|_{S^1} \right\|^2_{\mathbf{H}^{-3/2}_{S^1,h}} =$$

$$\frac{\sqrt{-1}}{2} \int\limits_{\mathbb{D}} \left(1-|z|^2\right)^2 |\psi(\overline{z}) - \psi_0(\overline{z})| \, dz \wedge \overline{dz} \leq$$

$$\left(\sup_{|z|<1} \left(1-|z|^2\right)^2 |\psi(\overline{z}) - \psi_0(\overline{z})| \right) \frac{\sqrt{-1}}{2} \int\limits_{\mathbb{D}} dz \wedge \overline{dz} < \frac{\delta}{2\pi}.$$

On the other hand we have

$$\left\| \left(\psi(\overline{z}) - \psi_0(\overline{z})\right)|_{S^1} \right\|^2_{\mathbf{H}^{-3/2}_{S^1,h}} = \left\| \Psi(\overline{z}) - \Psi_0(\overline{z}) \right\|^2_{\mathbf{H}^{3/2}_{S^1,h}} < \frac{\delta}{2\pi}.$$

Lemma 54 is proved since $\varepsilon = \frac{\delta}{2\pi}$. ■

According to Lemma 54 $\mathbb{H}^{1,1}_1(\mathbb{D})$ is an open set in $T^{3/2}_{id,\mathbf{T}^\infty}$. Nag proved that the map $\iota : \mathbb{H}^{1,1}_1(\mathbb{D}) \rightarrow \mathbb{HS}\left(\mathbf{H}^+, \mathbf{H}^-\right)$ is an isometric embedding when restricted on all μ_ψ such that $\psi(\overline{z})|_{S^1} \in C^\infty(\mathbb{C})$ with respect to the left invariant metrics. According to Lemma 54 $\mathbb{H}^{1,1}_1(\mathbb{D})$ is an open set in $T^{3/2}_{id,\mathbf{T}^\infty}$. Thus the linear map ι is an isometry and one to one on an open and everywhere dense subset in $\mathbb{H}^{1,1}_1(\mathbb{D})$ into the Hilbert space $\mathbb{HS}\left(\mathbf{H}^+, \mathbf{H}^-\right)$. This implies that ι is a continuous map from the open set $\mathbb{H}^{1,1}_1(\mathbb{D})$ of the Hilbert space in $\mathbb{H}^{1,1}(\mathbb{D})$ into the Hilbert space $\mathbb{HS}\left(\mathbf{H}^+, \mathbf{H}^-\right)$. So ι will be a continuos linear map of the Hilbert space $\mathbb{H}^{1,1}(\mathbb{D})$ into the Hilbert space $\mathbb{HS}\left(\mathbf{H}^+, \mathbf{H}^-\right)$. Theorem 49 is proved. ■

4.6 Hilbert $3/2$ Manifold Structure on the $\mathbf{T}^\infty$ and the Exponential Map

Definition 55 *We recall that a submanifold S of a Riemannian manifold M is called totally geodesic at p if each M-geodesic passing through p in a tangent direction to S remains in S for all time. If S is geodesic at all its points, then it is called totally geodesic [8].*

Theorem 56 was proved in [19]. We reproduced the proof of Theorem 56 in the Appendix of the paper.

Theorem 56 *Let $\psi \in \mathbb{HS}(\mathbf{H}^+, \mathbf{H}^-) = T_{\mathbf{H}^+,\mathbb{G}r_\infty}$. Let $\exp$ be the map from $\mathbb{HS}(\mathbf{H}^+, \mathbf{H}^-) = T_{H_+,\mathbb{G}r_\infty}$ to $\mathbb{G}r_\infty$ defined by the left invariant metric. Then the complex curve $\gamma_\psi(s) = \exp(s\psi) \subset \mathbb{G}r_\infty$ is a totally geodesic complex submanifold of complex dimension one in $\mathbb{G}r_\infty$ and $\gamma_\psi(s) = \exp(s\psi)$ exists for all $s \in \mathbb{C}$.*

The following analogue of the infinite dimensional complex analytic analogue of Hadamard's theorem was proved in [19] based on Theorem 56:

Theorem 57 *The complex exponential map defined in Theorem 56 is a covering complex analytic one to one map from the tangent space $T_{id,\mathbb{G}r_\infty}$ onto $\mathbb{G}r_\infty$. We reproduced the proof of Theorem 57 in the Appendix of the paper.*

Based on Theorems 56 and 57 by using very easy and standard arguments we will give a very simple proof of the following Theorem:

Theorem 58 *The completion $\mathbf{T}^{3/2}$ of $\mathbf{T}^\infty$ with respect to the left invariant Kähler metric defined by (15) is complex analytic Hilbert manifold isomorphic to the totally geodesic closed Hilbert submanifold $\exp\left(\iota\left(\mathbb{H}^{-1,1}(\mathbb{D})\right)\right)$ in $\mathbb{G}r_\infty$, where $\iota : \mathbb{H}^{-1,1}(\mathbb{D}) \to \mathbb{HS}\left(\mathbf{H}^+, \mathbf{H}^-\right)$ is the embedding defined in Theorem 49.*

Proof: A Theorem of Nag proved in [14] that $\mathbf{T}^\infty$ with the left invariant Kähler metric is isometrically embedded $\mathbb{G}r_\infty$. For each complex direction

$$\psi \in \left(\mathbf{H}^{3/2}_{S^1,h}\right) \approxeq \iota\left(\mathbb{H}^{-1,1}(\mathbb{D})\right) \subset T_{\mathbf{H}^+,\mathbb{G}r_\infty} = \mathbb{HS}(\mathbf{H}^+, \mathbf{H}^-)$$

the exponential map defines a complex analytic isomorphism between the complex line $s\psi$ and totally geodesic one dimensional complex submanifold

$$D_\psi := \gamma(s) = \exp\left(s\psi\right) \subset \mathbb{G}r_\infty$$

in the direction ψ. According to Theorem 49 $\iota\left(\mathbb{H}^{-1,1}(\mathbb{D})\right)$ is a closed Hilbert subspace in $T_{\mathbf{H}_+,\mathbb{G}r_\infty}$. Thus Theorem **??** implies that $\mathbf{T}^{3/2} := \exp\left(\iota\left(\mathbb{H}^{-1,1}(\mathbb{D})\right)\right)$ is a closed totally geodesic complex analytic submanifold in $\mathbb{G}r_\infty$ containing the image of $\mathbf{T}^\infty$ in $\mathbb{G}r_\infty$. According to Cor. 25 the Hilbert space $\mathbb{H}^{-1,1}(\mathbb{D})$ is isomorphic to the Hilbert space $\mathbf{H}^{3/2}_{S^1,h}$. Theorem 58 is proved. ■

Corollary 59 *The complex analytic Hilbert structure on $\mathbf{T}^{3/2}$ is modeled by the Sobolev $3/2$ space $\mathbf{H}^{3/2}_{S^1,h}$.*

Proof: Corollary follows directly from Theorem 22 and Theorem 58. ■

Definition 60 *By definition the infinite-dimensional Siegel disc is the set of Hilbert-Schmidt operators $T : \mathbf{H}^+ \to \mathbf{H}^-$ are such that $\det(I - TT^*) > 0$.*

Corollary 61 $\mathbf{T}^{3/2} \approxeq \exp\left(\iota\left(\mathbb{H}^{-1,1}(\mathbb{D})\right)\right)$ *is a totally geodesic complex analytic Hilbert submanifold isomorphic to the infinite-dimensional Siegel disc defined by the Hilbert-Schmidt operators $T \in \iota\left(\mathbb{H}^{-1,1}(\mathbb{D})\right) \subseteq \mathbb{HS}\left(\mathbf{H}^+, \mathbf{H}^-\right)$.*

Proof: Theorems 56, 57 and the definition of the complex analytic exponential map imply that $\exp\left(\iota\left(\mathbb{H}^{-1,1}(\mathbb{D})\right)\right)$ is a totally geodesic closed complex analytic Hilbert submanifold in $\mathbb{G}r_\infty$. We proved that any two points in both spaces can be joined by a unique geodesic and $\exp\left(\iota\left(\mathbb{H}^{-1,1}(\mathbb{D})\right)\right)$ is geodesically complete. This implies that $\exp\left(\iota\left(\mathbb{H}^{-1,1}(\mathbb{D})\right)\right)$ can be isometrically identified with the infinite-dimensional Siegel disc defined by the Hilbert-Schmidt operators $T \in \iota\left(\mathbb{H}^{-1,1}(\mathbb{D})\right)$. ■

Corollary 62 *The space* $\mathbf{T}^{3/2} := \left(\mathbf{Diff}_+^\infty(S^1)/\mathbb{PSU}_{1,1}\right)^{3/2}$ *equipped with the unique invariant Kähler metric has negative curvature in holomorphic directions. More precisely, the curvature is negative and uniformly bounded away from zero in holomorphic directions.*

Proof: According to the Theorem of Nag proved in [14], the embedding of $\mathbf{T}^\infty$ into the Grassmannian is isometric. Since $\mathbf{T}^\infty$ is an everywhere dense subset in $\mathbf{T}^{3/2}$, Theorem 49 implies the Hilbert manifold $\mathbf{T}^{3/2}$ is isometrically embedded into Segal-Wilson Grassmannian $\mathbb{G}r_\infty$. Cor. 62 followed directly from Theorem 48. ■

5 Appendix

5.1 Differential Geometry of the Segel Wilson Grassmannian

Theorem 48. *Let* $\{\psi_k\}$ *be an orthonormal basis of*

$$T_{\mathbf{H}^+}\mathbb{G}r_\infty = \mathbb{HS}(\mathbf{H}^+,\mathbf{H}^-) \subset Hom\left(\mathbf{H}^+,\mathbf{H}^-\right).$$

Then,

a. *The left invariant metric defined by* (71) *is Kähler and the component of the curvature tensor with respect to the orthonormal basis* $\{\psi_k\}$ *is given by:*

$$R_{i\bar{j},k\bar{l}} = -\delta_{i\bar{j}}\delta_{k\bar{l}} - \delta_{i\bar{l}}\delta_{k\bar{j}} + Tr\left(\psi_i\psi_j^* \wedge \psi_k\psi_l^*\right) + Tr\left(\psi_i\psi_l^* \wedge \psi_k\psi_j^*\right)$$

for $(i,j) \neq (k,l)$ *and* $R_{i\bar{j},i\bar{j}} = -2\delta_{i\bar{j}} + Tr\left(\psi_i\psi_j^* \wedge \psi_k\psi_l^*\right)$. **b.** *Let* ψ *be any complex direction in the tangent space* $T_{\mathbf{H}^+}\mathbb{G}r_\infty$. *Let* K_ψ *be the Gaussian sectional curvature in any the two-dimensional real space defined by* $\operatorname{Re}\psi$ *and* $\operatorname{Im}\psi$. *Then we have*

$$K\psi = -2 + Tr\left(\psi\psi^* \wedge \psi\psi^*\right) < -\frac{3}{2}.$$

Proof of part a: The proof is based on the construction of the so-called Cartan coordinate system in [13]. By that we mean a holomorphic coordinate system $(x^1, x^2, \dots)$ in which the components of the Kähler metric tensor $g_{i\bar{j}}(x)$ is given by the formula: $g_{i\bar{j}} = \delta_{i\bar{j}} + r_{i\bar{j},i\bar{j}}x^k\bar{x}^l + \mathcal{O}(|x|^3)$. Cartan proved that if the coordinate system satisfies this last equation, then $r_{i\bar{j},i\bar{j}} = -\frac{1}{3}R_{i\bar{j},i\bar{j}}$. where $R_{i\bar{j},k\bar{l}}$ is the curvature tensor. To prove part **a)** we construct the Cartan coordinates in $\mathbb{G}r_\infty$:

Definition A.48.1 *Let* ψ_i *be an orthonormal basis of* $\mathbb{HS}(\mathbf{H}^+,\mathbf{H}^-)$. *Let*

$$\varphi_t = \sum_{i=1}^{\infty} t_i\psi_i \in \mathbb{HS}(\mathbf{H}^+,\mathbf{H}^-).$$

Let the subspace $\mathbf{W}_t$ *of* $\mathbf{H}^+ \oplus \mathbf{H}^-$ *be spanned by the set*

$$\{1 + \varphi_t(1), z + \varphi_t(z), \dots, z^n + \varphi_t(z^n), \dots\}.$$

Obviously, $\mathbf{W}_t$ *is the graph of the operator* φ_t*. We define the exponential map*

$$\exp : \mathbb{HS}(\mathbf{H}^+,\mathbf{H}^-) \to \mathbb{G}r_\infty$$

as follows

$$\exp\left(\sum_{i=1}^{\infty} t_i\psi_i\right) = \mathbf{W}_t \in \mathbb{G}r_\infty. \tag{75}$$

Thus $t = (t_1, t_2, \dots)$ *will define the local coordinates in some open set* $U \subset \mathbb{G}r_\infty$ *of the point* 0 *corresponding to* $\mathbf{H}^+ \in \mathbb{G}r_\infty$*.*

Lemma A.48.2 *The following expansion near* $t = 0$ *in the coordinates* $t = (t_1, t_2, \dots)$ *holds*

$$-\frac{\partial^2}{\partial t^i \overline{\partial t^j}} \log\det\left(1 - \sum_{i,j} t^i\overline{t^j}\psi_i\psi_j^*\right) =$$

$$\delta_{i\bar{j}} + \left(2\delta_{i\bar{j}} - Tr\left(\psi_i\psi_j^* \wedge \psi_j\psi_i^*\right)\right) t^i\overline{t^j} +$$

$$\sum_{(i,j)\neq(k,l)} \left(\delta_{i\bar{j}}\delta_{k\bar{l}} + \delta_{i\bar{l}}\delta_{k\bar{j}} - Tr\left(\psi_i\psi_j^* \wedge \psi_k\psi_l^*\right) - Tr\left(\psi_i\psi_l^* \wedge \psi_k\psi_j^*\right)\right) t^k\overline{t^l}.$$

Proof: Let $f(t) = \det\left(id - \sum t^i\overline{t^j}\psi_i\psi_j^*\right)$, then

$$\frac{\partial^2}{\partial t^i \overline{\partial t^j}} \log\det\left(id - \sum t^i\overline{t^j}\psi_i\psi_j^*\right) = \frac{\partial^2}{\partial t^i \overline{\partial t^j}} f^{-1} - \frac{\partial f}{\partial t^i}\frac{\overline{\partial} f}{\overline{\partial t^j}} f^{-2}. \tag{76}$$

From the definition of the determinant, we have that

$$\det\left(id - \sum_{i,j} t^i\overline{t^j}\psi_i\psi_j^*\right) =$$

$$1 - \sum_{i,j} \mathrm{Tr}(\psi_i\psi_j^*)t^i\overline{t^j} + \sum_{i,j,k,l} \mathrm{Tr}(\psi_i\psi_j^* \wedge \psi_k\psi_l^*)t^i\overline{t^j}t^k\overline{t^l} + \text{h.o.t.}$$

Hence,

$$\frac{\partial^2}{\partial t^i \overline{\partial t^j}} \det\left(id - \sum t^i\overline{t^j}\psi_i\psi_j^*\right) = -\delta_{i\bar{j}} + \mathrm{Tr}(\psi_i\psi_j^* \wedge \psi_i\psi_j^*))t^i\overline{t^j} +$$

$$\sum_{(k,l)\neq(i,j)} \mathrm{Tr}\left(\psi_i\psi_j^* \wedge \psi_k\psi_l^* + \psi_i\psi_l^* \wedge \psi_k\psi_j^*\right) t^k\overline{t^l} + \text{h.o.t},$$

$$\frac{\partial}{\partial t^i} \det\left(id - \sum_{i,j} t^i\overline{t^j}\psi_i\psi_j^*\right) = -\sum_l \mathrm{Tr}(\psi_i\psi_l^*)\bar{t}_l + \text{h.o.t.},$$

and
$$\frac{\overline{\partial}}{\overline{\partial t^j}} \det\left(id - \sum t^i \overline{t^j} \psi_i \psi_j^*\right) = -\sum_k \mathrm{Tr}(\psi_k \psi_j^*) t^k + \text{h.o.t.}.$$
Furthermore,
$$\frac{1}{\det\left(id - \sum t^i \overline{t^j} \psi_i \psi_j^*\right)} = 1 + \sum t^i \overline{t^j} \mathrm{Tr}(\psi_i \psi_j^*) + \text{h.o.t}$$
and
$$1/f^2 = 1 + 2\sum t^i \overline{t^j} \mathrm{Tr}(\psi_i \psi_j^*) + \text{h.o.t.}$$
Substituting the last 5 equations into equation (76) we obtain the result. Lemma **A.48.2** is proved. ■

Lemma A.48.2 implies that the coordinates $(t_1, t_2, \dots)$ forms a Cartan coordinate system. Thus, for $(i,j) \neq (k,l)$ we get:
$$R_{i\bar{j},k\bar{l}} = -\delta_{i\bar{j}}\delta_{k\bar{l}} - \delta_{i\bar{l}}\delta_{k\bar{j}} + \mathrm{Tr}(\psi_i\psi_j^* \wedge \psi_k\psi_l^* + \psi_i\psi_l^* \wedge \psi_k\psi_j^*),$$
and
$$R_{i\bar{j},i\bar{j}} = -2\delta_{i\bar{j}} + \mathrm{Tr}(\psi_i\psi_j^* \wedge \psi_i\psi_j^*).$$
So we proved

Corollary A.48.3 *The left invariant metric on $\mathbb{G}r_\infty$ is Kähler with a potential* $\log\det\left(id - \sum t^i \overline{t^j} \psi_i \psi_j^*\right)$.

This concludes the proof of **part a** of Theorem 48. ■.

Proof of part b: To prove part b) remark that the Gaussian curvature in the direction ψ_i is given by
$$K_{\psi_i} = R_{i\bar{j},i\bar{j}} = -2 + \mathrm{Tr}(\psi_i\psi_i^* \wedge \psi_i\psi_i^*).$$
Now, if $\psi_1 = \varphi$, we complete the set $\{\psi_1\}$ to an orthonormal set in the Hilbert space $T_{H_+,\mathbb{G}r_\infty}$. Hence, from the previous discussion it follows that $K_\varphi = -2 + \mathrm{Tr}(\varphi\varphi^* \wedge \varphi\varphi^*)$.

Lemma A.48.4 *If $Tr\left(\psi\psi^*\right) = 1$, then $Tr\left(\wedge^2\psi\psi^*\right) < \frac{1}{2}$.*

Proof: Note that $\varphi\varphi^*$ is a compact positive operator and hence its nonzero eigenvalues are all positive. Let $\{\lambda_i\}$ be the set of nonzero eigenvalues of $\varphi\varphi^*$. Since $\varphi\varphi^*$ is trace-class and $||\varphi||^2 = \mathrm{Tr}\left(\varphi\varphi^*\right) = 1$ imply that
$$\mathrm{Tr}\left(\varphi\varphi^*\right) = \sum_i \lambda_i = 1.$$
A simple argument with tensor products gives
$$Tr(\wedge^2\left(\varphi\varphi^*\right)) = \sum_{i<j} \lambda_i\lambda_j.$$
We have
$$1 = \left(\sum_i \lambda_i\right)^2 = 2\sum_{i<j}\lambda_i\lambda_j + \sum_i \lambda_i^2.$$

From here it follows that $Tr(\wedge^2\varphi\varphi^*) < 1/2$. Using the formula for the curvature in holomorphic direction φ it follows that $K_\varphi < -3/2 < 0$. ■ **Lemma A.48.4** implies part **2** of Theorem 48. Theorem 48 is proved. ■

Definition A.48.5 *We recall that a submanifold S of a Riemannian manifold* M *is called totally geodesic at p if each* M*-geodesic passing through p in a tangent direction to S remains in S for all time. If S is geodesic at all its points, then it is called totally geodesic* [8].

Theorem 56 was proved in [19]. We reproduce the proof. The reason is to make this paper self contained.

Theorem 56. *Let $\psi \in \mathbb{HS}(\mathbf{H}^+, \mathbf{H}^-) = T_{\mathbf{H}^+, \mathbb{G}r_\infty}$. Let* exp *be the map from $\mathbb{HS}(\mathbf{H}^+, \mathbf{H}^-) = T_{H_+, \mathbb{G}r_\infty}$ to $\mathbb{G}r_\infty$ defined by* **Definition A.48.1** *Then the complex curve $\gamma_\psi(s) = \exp(s\psi) \subset \mathbb{G}r_\infty$ is a totally geodesic complex submanifold of complex dimension one in $\mathbb{G}r_\infty$ and $\gamma_\psi(s) = \exp(s\psi)$ exists for all $s \in \mathbb{C}$.*

Proof: The proof of the Theorem 56 is a consequence of the following two results:

Lemma A.56.1 *Let $\psi \in \mathbb{HS}\left(\mathbf{H}^+, \mathbf{H}^-\right)$ be such that $\|\psi\|^2_{HS} = 1$ and*

$$s(t) = s_0 + te^{i\theta},$$

with $\theta, t \in \mathbb{R}$ be a line in $\mathbb{R}$. Let $\gamma_\psi(s(t)) := \exp(s(t)\psi)$ be a curve in $\mathbb{G}r_\infty$. Then, the norm of the tangent vector $\dot{\gamma}_\psi$ to the the path $\gamma_\psi(s(t))$ has a length one, i.e. $\|\dot{\gamma}_\psi(s(t))\| = 1$.

Proof: By the definition of the exponential map the line $s(t)\psi$ in the Hilbert space $\mathbb{HS}\left(\mathbf{H}^+, \mathbf{H}^-\right)$ corresponds to the family of Hilbert subspaces $\mathbf{W}_s := \gamma_\psi(s) \subset \mathbf{H}^+ \oplus \mathbf{H}^-$ in $\mathbb{G}r_\infty$ spanned by the vectors

$$s(t)\psi(1), ..., s(t)\psi(z^n), ...$$

With respect to the decomposition $\mathbf{H} = \mathbf{H}^+ \oplus \mathbf{H}^-$ define the block matrix

$$A_s = \begin{pmatrix} id & \overline{s}\psi^* \\ s\psi & id \end{pmatrix}$$

From the definition of $\mathbf{W}_s$ we have $\mathbf{W}_s = A_s\mathbf{H}^+$. Let

$$T : \mathbf{H}^+ \oplus \mathbf{H}^- \rightarrow \mathbf{H}^+ \oplus \mathbf{H}^-$$

be any bounded operator. Using that $Graph(T)$ is perpendicular to $Graph'(T^*)$, where

$$Graph'(T) \overset{def}{=} \{(-Tx, x) \ |x \in Dom(T)\ \},$$

we get $\mathbf{W}_s^\perp = A_s\mathbf{H}^-$. Hence, the matrix A_s maps $\mathbf{H} = \mathbf{H}^+ \oplus \mathbf{H}^-$ into $\mathbf{W}_s \oplus \mathbf{W}_s^\perp$ and preserves the direct sum decomposition. It is well known that the operators $(id + s\overline{s}\psi\psi^*)$ and $(id + s\overline{s}\psi^*\psi)$ are invertible. The following relations can be checked easily:

$$(id + s\overline{s}\psi^*\psi)^{-1}\psi^* = \psi^*(id + s\overline{s}\psi\psi^*)^{-1}, \tag{77}$$

and

$$(id + s\overline{s}\psi\psi^*)^{-1}\psi = \psi(id + s\overline{s}\psi^*\psi)^{-1}. \tag{78}$$

Therefore,

$$A_s^{-1} = \begin{pmatrix} id & \overline{s}\psi^* \\ s\psi & id \end{pmatrix} \begin{pmatrix} (id + s\overline{s}\psi^*\psi)^{-1} & 0 \\ 0 & (id + s\overline{s}\psi\psi^*)^{-1} \end{pmatrix} =$$

$$\begin{pmatrix} (id + s\overline{s}\psi^*\psi)^{-1} & 0 \\ 0 & (id + s\overline{s}\psi\psi^*)^{-1} \end{pmatrix} \begin{pmatrix} id & \overline{s}\psi^* \\ s\psi & id \end{pmatrix}. \tag{79}$$

To compute the norm $||\dot{\gamma}_\psi(s(t))||$, we use the identification of $T_{\mathcal{W}_{s(t)},\mathbb{G}r_\infty}$ with $\mathbb{HS}(\mathbf{W}_{s(t)}, \mathbf{W}_{s(t)}^\perp)$. The latter, is mapped onto $\mathbb{HS}(\mathbf{H}^+, \mathbf{H}^-)$ by means of $\widetilde{X} \mapsto A_s^{-1}\widetilde{X}A_s\,|_{\mathbf{H}^+}$. From the invariance of the metric we have that the norm of $\widetilde{X} \in T_{\mathbf{W},\mathbb{G}r_\infty}$ is given by

$$||\widetilde{X}||^2 = \mathrm{Tr}\left((A_s^{-1}\widetilde{X}A_s\,|_{\mathbf{H}^+})^*(A_s^{-1}\widetilde{X}A_s\,|_{\mathbf{H}^+}\,.)\right). \tag{80}$$

Since,

$$\dot{\gamma}_\psi(s(t)) = \begin{pmatrix} 0 & -e^{-i\theta}\psi^* \\ e^{i\theta}\psi & 0 \end{pmatrix},$$

a technical but straightforward computation with $\widetilde{X} = \dot{\gamma}_\psi(s(t))$ taking into account equations (77), (78), and (79) yields the Lemma **A.56.1**. ■

Lemma A.56.2 *Set* $\dot{\gamma}_\psi(s) = \frac{d}{ds}\gamma_\psi(s)$ *and assume that* $\|\dot{\gamma}_\psi(s)\| = 1$*. Let* ∇ *be the covariant derivative in* $\mathbb{G}r_\infty$ *given by the Levi-Civita connection. Then,*

$$\nabla_{\dot{\gamma}_\psi(s)}\left(\dot{\gamma}_\psi(s)\right) = 0.$$

Proof: Let $s \in \mathbb{R}$ and $\gamma_\psi(s)$ be a geodesic in our Kähler manifold. For each point s of the geodesic $\gamma_\psi(s)$ we define a complex direction as follows:

$$\dot{\gamma_\psi}(s) + i\mathcal{I}\dot{\gamma_\psi}(s).$$

For each s in $\gamma_\psi(s)$ we consider a geodesic $\mathcal{I}\dot{\gamma_\psi}(s)$ and so each point τ on the geodesic from s with direction $\mathcal{I}\dot{\gamma_\psi}(s)$ we have two tangent vectors. The first, $\alpha(\tau)$, which is the parallel transport of $\dot{\gamma_\psi}(s)$ and the second $\beta(\tau)$ which is given by

$$\beta(\tau) = \frac{d}{ds}\mathcal{I}\dot{\gamma_\psi}(s),$$

and is the parallel transport of $\mathcal{I}\dot{\gamma_\psi}$. The Kähler condition implies that the complex structure operator is parallel with respect to the Levi-Civita connection of the metric. From here we get that for the Lie bracket $[\dot{\gamma}(s), \mathcal{I}\dot{\gamma}(s)]$ of the vector fields we have $[\dot{\gamma}(s), \mathcal{I}\dot{\gamma}(s)] = 0$. Hence, it follows from Frobenius theorem that there exist a surface S such that the tangent space is spanned by $\alpha(s)$ and $\beta(s)$. Since,

$$\mathcal{I}(\alpha(s) + i\beta(s)) = i(\alpha(s) + i\beta(s))$$

it follow that S is a complex analytic curve and we can take z as a complex analytic coordinate associated to the point

$$z = \exp(x\dot{\gamma_\psi}(s) + iy\mathcal{I}\dot{\gamma_\psi}(s)).$$

Let's write $\dot{\mu}(z) = \dot{\gamma_\psi}(s) + i\mathcal{I}\dot{\gamma_\psi}(s)$. From the properties of the Levi-Civita connection and $\|\dot{\mu}(z)\|^2 = const$ it follows that

$$0 = \frac{d}{dz}||\dot{\mu}(z)||^2\,|_{s_0} = 2\langle \nabla_z \dot{\mu}(z), \dot{\mu}(z)\rangle = 0, \tag{81}$$

and so

$$\frac{d^2}{d\bar{z}dz}||\dot{\mu}(z)||^2\,|_{s_0} = ||\nabla_z \dot{\mu}(z)||^2 - R_\psi ||\dot{\mu}(z)||^2 = 0.$$

Since $R_\psi < 0$, we have $\nabla_z \dot{\mu}(z) = 0$. Restrict $\nabla_z \dot{\mu}(z)$ on the real part at s_0 and we get that $\nabla_z \dot{\gamma_\psi}(z) = 0$. Hence, taking $\mathrm{Re} z = \dot{\gamma_\psi}$, we have $\nabla_{\dot{\gamma_\psi}(s)} \dot{\gamma_\psi}(s)\,|_{s_0} = 0$. Lemma **A.56.2** is proved. ■ Theorem 56 is proved. ■

In this section we will prove the infinite dimensional complex analytic analogue of Hadamard's theorem, to show the existence of the complex analytic manifold structure $\mathbb{H}^{-1,1}(\mathbb{D})$ on $\mathbf{T}^\infty$.

Theorem 57. *The complex exponential map defined in Theorem 56 is a covering complex analytic one to one map from the tangent space $T_{id,\mathbb{G}r_\infty}$ onto $\mathbb{G}r_\infty$.*

Proof: The proof Theorem 57 is based on Theorem 56.

Lemma A.57.1 *Let $\psi \in T_{\mathbf{H}^+,\mathbb{G}r_\infty} = \mathbb{HS}\left(\mathbf{H}^+, \mathbf{H}^-\right)$ and $\|\psi\|_{HS} = 1$. Then the exponential map restricted to one dimensional complex subspace*

$$\{c\psi | c \in \mathbb{C},\ \psi \in \mathbb{HS}\left(\mathbf{H}^+,\mathbf{H}^-\right)\} \subset T_{\mathbf{H}^+,\mathbb{G}r_\infty} = \mathbb{HS}\left(\mathbf{H}^+,\mathbf{H}^-\right)$$

and taking values in the totally geodesic submanifold D_ψ is a complex analytic diffeomorphisms.

Proof: To prove Lemma **A.57.1** we use that the curvature in the holomorphic direction is negative. This fact implies that we have

$$||\left(d\exp(c\overrightarrow{w})\right)(\overrightarrow{w})\,|| \geq ||\overrightarrow{w}||. \tag{82}$$

For the proof see [8]. From a standard result in (finite dimensional) differential geometry states that (82) implies that the map is a local diffeomorphisms and thus it is covering. Furthermore, if $c_1 \neq c_2$ then

$$\mathbf{W}_{c_1\psi} := (id + c_1\psi)\,(H_+) \neq \mathbf{W}_{c_2\psi} := (id + c_2\psi)\left(\mathbf{H}^+\right).$$

Hence, $\exp(c_1\psi) \neq \exp(c_2\psi)$. Thus the exponential map

$$\exp : c\psi \to \exp(c\psi) = D_\psi$$

is a diffeomorphisms. Next we will show that it is complex analytic map. We need to show that if $\overrightarrow{v} \in T_{\exp(c_1\psi),D_\psi}$, then $J(\overrightarrow{v}) \in T_{\exp(c_1\psi),D_\psi}$, where J is the complex structure operator. $\overrightarrow{v}$ is obtained by a parallel transform of a vector $\overrightarrow{v}_0 \in T_{0,D_\psi}$ along the total geodesic submanifold D_ψ. We know that $J(\overrightarrow{v}_0) \in T_{0,D_\psi}$. Since the metric is Kähler then $\nabla J = 0$. Since D_ψ is a totally geodesic submanifold, then by the parallel transposition $\overrightarrow{v}_0 \in T_{0,D_\psi}$ goes to $\overrightarrow{v} \in T_{\exp(c_1\psi),D_\psi}$. So $\nabla J = 0$ and D_ψ is a totally geodesic submanifold imply

that by parallel transport of $J(\overrightarrow{v}_0) \in T_{0,D_\psi}$ goes to $J(\overrightarrow{v}) \in T_{\exp(c_1\psi),D_\psi}$. Lemma **A.57.1** is proved. ■

Suppose that ψ_1 and ψ_2 are linearly independent vectors. Then the construction of exp and (82) imply that complex curves $\{\exp(t\psi_1)|t \in \mathbb{C}\}$ and $\{\exp(t\psi_2)|t \in \mathbb{C}\}$ intersect only at the identity. These arguments imply that exp is a complex analytic isomorphism of some open set $T_{id,\mathbb{G}r_\infty}$ to an open set in $\mathbb{G}r_\infty$. Since the curvature of the Kähler metric is negative in complex direction then (82) implies that the exponential map is surjective. Theorem **??** is proved. ■

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